Project on the development of a MATLAB algorithm for the calculation of dynamic response of rotatory structures based on finite elements

REPORT

June 2015
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1. Introduction

1.1. Objective

The objective of this study is the transformation of an existing in-house MATLAB code based on Finite Element Method that calculates dynamic behavior of non-rotating structures. The transformation must make the code able to calculate the rotational dynamics of rotating structures, mainly the eigenvalues and eigenvectors, given the rotational speed and the axis of rotation. These calculations are useful for structures such as motors, wind turbines, railway axes…

After modifying the code, results obtained in fixed structures and rotating structures have to be compared, in order to analyze which is the effect of a rotational speed in the vibration response of a rotating structure. Moreover, this project is a work-package of a more global project that has the objective of implementing an in-house FEM/BEM-based solver for the calculation of the noise radiation of such structures in detail.

1.2. Scope

As it has been explained, this project is part of a more global project. Furthermore, a part of this bigger code has already been done. This existing in-house part of the code is capable to calculate the required dynamic behavior, but only in non-rotating structures. The task of assembling Finite Element Method matrices and extracting the information of the meshes does not have to be carried out. Hence, the main task to be performed is modifying the code to make it able to calculate the dynamics of rotatory structures: it can be done taking advantage of the existing matrices, which means modifying them to account for the rotation of the structure. Despite this, the information of the nodes and elements must be introduced to the MATLAB code in each studied case, so new meshes should be created (using a preprocessor software, such as GID or Patran).

The objective of the bigger project, which is being implemented by researchers of the LEAM (Laboratory of Acoustical and Mechanical Engineering) in Terrassa, is the calculation of noise radiation of rotating structures in detail. This calculation needs the study of fluid-structure interaction, wave propagation in the air and other technical issues, but the scope of the part of the code done in this project ends with the solving of structural rotational problems.

1.3. Requirements

In order to achieve the objectives, the developed MATLAB code should have some characteristics and capabilities. The MATLAB code must be able to:

- Solve the dynamic response of rotating structures. Rotating eigenvectors and eigenvalues are the most important parameters for the study. These values give information of the free response of a rotating structure, which is a first step in order to calculate the noise radiation.

- Be efficient computationally. There is going to be thousands of nodes, in order to make accurate calculations. This means that FEM matrices are going to be very large, and the calculus could take too much time if the implementation is not efficient enough.

- Adapt to the needs of LEAM. This in-house code is being developed because the existing commercial codes for the analysis of rotating structures are not adequate for the LEAM specific studies. Therefore, the code for rotating structures should follow the structure of the existing one, explaining each new function in order to allow LEAM researchers understanding and modifying the code easily.

1.4. Background. Problem statement

The existing commercial codes for the study of noise radiation are not adequate for the studies that LEAM needs. These commercial codes are, in general, too generic, and because of this reason they do not work well in some specific cases. Thus, making an in-house code is very useful, because it can work with the specific requirements and features of the problems of the researchers of LEAM.

Despite this, there are some problems in the creation of a new code. One of the greatest troubles of this method is the difficulty of proving the results: not many problems about the dynamic response of rotating structures in 3D can be solved analytically. Another issue to take into account is the fact that the resolution of rotatory structures in 3D is probably going to have much computational load, especially if the structures are not simple. The meshes in complex 3D geometries have to be really accurate in some parts of the structures in order to obtain good results, and it means a lot of computational load. Therefore, the code should be optimized in order to keep the computational costs controlled.

The usefulness of the project is its application on real cases. The mechanical behavior and the vibration response of the structures are not the same when they are rotating and when they are fixed, so it is important to know the changes that appear. Furthermore, a non-rotational problem can be understood as a particular case of a rotational problem with rotational speed equal to zero, so the aim of this project is the generalization of the problem.

2. Development

2.1. State of the art

2.1.1. Finite Element Method

Finite Element Method (FEM from now) is one of the most powerful and used tools in a lot of engineering fields. Its development has supposed a major breakthrough for the engineering in the last 50 years. Nowadays it is common used in many engineering projects.

2.1.1.1. HISTORY

FEM has been a very important tool for engineering since 1950. At the beginning, it was used only with bar elements. K. Hrennikoff and R. Courant developed the bases of this method during the 1950's. FEM was developed in the 1960's because aerospace engineering needed a method to analyze continuous structures, such as surfaces and volumes. K. Feng participated in the development of FEM through the proposition of a systematic numerical method for solving partial differential equations. Researchers of the University of Stuttgart, the University of California (Berkeley), the University of Swansea, The University of Paris and the Cornell University helped J. H. Argyris, who was a pioneer in computer science, to establish the formulation of FEM during the 1960's and the 1970's (Wikipedia, 2015). The importance of FEM for aeronautical and aerospace engineering is so big that NASA sponsored the first version of NASTRAN, which nowadays is one of the most powerful software for structural analysis. G. Strang and G. Fix, both American mathematicians, provided a strong and rigorous mathematical basis to FEM in 1973 with a publication. After that, the method was generalized to make it useful for other fields of the science, such as electromagnetism, heat transfer or fluid dynamics (MIT, 2012).

2.1.1.2. CONCEPTUAL DESCPRITION

The aim of FEM is dividing a big problem into a lot of smaller problems. FEM divides a continuous structure, such a surface or a volume, into different pieces of the structure, which are called finite elements. This elements can be triangles or quadrilaterals in 2D, and tetrahedral or hexahedral in 3D, among other forms. This dividing process creates a mesh, formed by the elements. Each element has some nodes, which are related between them. The partition of the structure is useful because a continuous structure has infinite degrees of freedom (DOF), and a finite structure has finite DOF. Moreover, it is very difficult to make the algebraic equations for the whole domain. The result of dividing the structure is a set of simultaneous algebraic equations, and the problem can be solved through them.

Adjacent elements share the DOF at the connecting nodes. Each element has its own algebraic equations (or each node if nodal form of FEM is used). All of these equations are related because elements and nodes are linked, so a matrix system can be written and solved. The constituent matrices are computed through the shape functions, which are inherent in the method, the geometry and the properties of the material.

It is necessary to define the government equations and the boundary conditions in the problems that engineers want to solve. The government equations are given by the science in each case; some examples are the Hooke's Law and the Fourier's Law for the heat flux transfer, among others. The boundary conditions define the problem: depending on the type of problem they can be imposed displacements at some points, imposed temperatures…

It is important to notice that FEM provides an approximately solution, logically. The fact of dividing the problem into several smaller problems allows the engineers to get a good solution in a fast way, but it is not an empiric solution. Therefore, it is convenient to be careful with the results and to make sure that the used mesh is accurate enough for the studied problem, especially in critical zones such as edges, holes… On the other hand, using a mesh too much accurate creates additional computational load, and this is a big issue in engineering projects. More computational load means more time, and more time means more money. Hence, it is really important to select an adequate thickness for the mesh: it must be sufficiently accurate to give a good solution but not too much, in order not to create unnecessary computational load (de Weck & Kim, 2004).

2.1.1.3. ADVANTAGES AND DISADVANTAGES OF FEM

The main advantages of FEM are the fact that it can handle virtually any geometry, the fact that it can solve a large amount of engineering problems and the fact that it can provide a good solution in a relatively short time (depending on the case).

FEM has some disadvantages. First of all, solutions obtained using FEM are not exact solutions, but they can be very similar to the exact ones if the mesh refinement is adequate. Besides, there is not a method that provides exact solutions for problems with complex geometries, so FEM can be a good choice. In the second place, there is software that uses FEM, but it is not very user-friendly. This fact is due to two factors: firstly, the inherent complexity of FEM, which is a mathematically complicated method, and secondly, the fact that companies are not interested in making software more user-friendly, because they sell formation courses. Furthermore, FEM has inherent errors on his formulation; this formulation is not perfect because it takes some assumptions that are not always true.

Another disadvantage is the fact that FEM do not produce a closed-form solution: a closed-form solution would allow the user to analyze system response to changes in several parameters.

2.1.2. Boundary Element Method

Boundary Element Method (BEM from now) is a computational technique used in a lot of engineering fields, such as fluid mechanics and acoustics. BEM is based on solving linear partial differential equations formulated as integral equations in the boundaries of the systems. Thus, BEM reduces the meshing problem from volumes to surfaces, and it represents a huge advantage in many cases. For solving a problem using BEM, it is necessary to describe the Green's functions of the different medium in which the system consist of. Green's functions are explained below. Green's functions can always be calculated, but sometimes using them is not a good choice. Depending on the problem features, expressions for Green's functions can be non-analytical or semi-analytical expressions. If the expressions are not analytical, the process applying BEM can be slower than applying FEM, because the integration process can be difficult. It usually happens if mediums are not continuous and homogeneous, because these kinds of mediums usually have "simple" analytical expressions for Green's functions. In this aspect, although nonlinearities can be included in the model, they generally add volume integrals that require the volume to be discretized, so the main advantage of BEM is removed (Wikipedia, 2015).

BEM is useful complementing other methods as FEM. The coupling of these methods is common in a lot of engineering fields, because sometimes it is necessary to know what happens inside a structure (for example a tunnel) and what happens far away from the structure (for example the floor and the buildings). If BEM is used for modeling the ground, the number of nodes can be reduced a lot. The part of the code developed in this project does not use BEM, but the bigger code will use it for the study of noise radiation.

The internal part of the structure can be meshed in FEM and the external one (the environment) in BEM in order to reduce the number of nodes. BEM works faster than FEM if the ratio surface/volume is small. Although BEM is often more efficient than FEM, it is important to notice that the matrices in BEM are fully populated (FEM matrices are often almost tri-diagonal matrices). Hence, it is important to be careful when using BEM, because it is necessary to ensure that Green's functions can be calculated efficiently and that the ratio surface/volume is small: if these features are not true, the fact that BEM matrices are fully populated slows the method (Costabel, 1987).

2.1.2.1. GREEN'S FUNCTIONS

Green's functions of a medium define the response of this medium to an excitation located in an arbitrary position. In the time domain, a Green's function is the impulse response of the medium and, in the frequency domain, is the frequency response function of it. In time domain, initial conditions must be known to properly use the Green's functions. These functions are the key in BEM.

Each type of medium has its own Green's functions, which depend on the material properties and the geometric configuration of the medium itself. Homogeneous medium, such a not-moving fluid or an isotropic solid have analytical Green's functions in both time and frequency domains, which simplifies a lot the integration process of them along the boundaries. In more complicated mediums, such layered isotropic media, the computation of the Green's functions and its integration along the boundaries become a more time consuming task, but normally still more efficient than using FEM (Costabel, 1987).

2.1.2.2. ADVANTAGES AND DISADVANTAGES OF BEM

BEM could have important advantages in computational effort against other methods, but it has other problems. Among other advantages, in BEM only the boundary of the domain has to be discretized, instead of all the volume. This can mean a huge time reduction when solving problems. Besides, in some problems the relevant data are on the boundary, not inside the domain. The boundary values obtained using FEM are often inaccurate, so using BEM is usually a better choice, because the solution of BEM boundary integral equations provides a good solution.

On the other hand, BEM has some problems and disadvantages. First of all, it is difficult to obtain the solution close to the boundary using BEM. Secondly, boundary integral equations require the explicit knowledge of a fundamental solution of the differential equation, which is sometimes difficult to obtain.

Generally, problems with non-homogeneous medium or nonlinear differential equations cannot be solved using basic BEM. As it has been explained before, expressions in this type of problems are not analytical, and this is an important issue. This kind of problems can usually be achieved coupling FEM and BEM. Another disadvantage of BEM is the fact that for a given boundary value problem there are different equations and several numerical approximation methods. This means that BEM has to evaluate all the possibilities, which means more time.

Moreover, it is difficult to analyze mathematically the boundary integral equations, because sometimes they are not ordinary Fredholm integral equations of the second kind. In addition, sometimes there are non-integrable singularities in the domain, and then some assumptions have to be taken. These singularities can appear if the boundary has corners or edges, or if the boundary conditions are discontinuous. BEM can treat these singularities more directly than FEM (because these singularities are on the boundary) but they are difficult to analyze. In fact, using BEM in 3D domains with corners and edges is nowadays an incomplete technique and it needs to be studied and developed. These singularities have been well solved in 2D problems (Boundary Element Method Web, 2015).

2.1.3. MSC Software

MSC Software has developed software that provides solutions for rotating structures, such as tail rotor shafts, turbo-expanders or driven units. This software can address a lot of rotating problems. It can compute if the structure is capable of sustaining operating conditions, what loads are acting on the moving structure, how long the structure will last under repeated loading (this is a very important aspect for rotating structures, because fatigue is usually critical in this kind of structures) and if the acoustics of the structure are adequate. It is possible to understand the structural behavior of the structures using this software, because it is able to make both static and rotordynamic analysis. Specifically, it can be done with MSC Nastran (MSC Software, 2015).

2.1.3.1. MSC NASTRAN

Through the static analysis, it is possible to determine loads on rotor, the stresses and the displacements. Through the rotordynamic analysis, it is possible to compute complex eigenvalues, critical speeds, rotor and support responses to arbitrary excitation and transient response, with linear or nonlinear models. Besides, MSC Nastran has some other capabilities in the field of the rotordynamic analysis. These capabilities are the following: modeling multiple rotor systems, performing random vibration analysis or simulating contact of multiple bodies, among others (MSC Software, 2015).

2.1.3.2. MSC NASTRAN EMBEDDED FATIGUE AND MSC FATIGUE

Using MSC Nastran Embedded Fatigue (NEF) or MSC Fatigue is possible to know the life of products or structures. MSC Nastran Embedded Fatigue can compute the stress life and the strain life. It can process up to 100 threads in parallel and can analyze multiple fatigues in a single job submission.

MSC Fatigue can compute the stress life and the strain life too, but it can also implement linear elastic fracture mechanics methods. This software has no limit on number of nodes or elements analyzed, has compliance function library including numerous crack geometries and is able to analyze static, quasi-static and transient loading. It can analyze spot welds and can support simultaneous application of up to 500 load cases. Furthermore, it has support for RPC, DAC and ASCII load files, among other capabilities (MSC Software, 2015).

2.1.3.3. PATRAN

This software has access to CAD geometry and full suite of meshing tools. Patran has geometric feature recognition/modification with associated mesh. Moreover, it can provide support for third-party solvers, such as Abaqus, Ansys or LS-Dyna, and it has complete post-processing capabilities. It can also work as a preprocessor (MSC Software, 2015).

2.1.3.4. ADAMS

Adams allows the user to determine mechanical loads, and it is a powerful numerical analysis tool that solves equations of motion for a lot of kinds of dynamic simulations, such as the kinematic, the static or the quasi-static ones. It is useful for automating the simulation process and for customizing the interface. It can also analyze forced vibrations (MSC Software, 2015).

2.1.3.5. ACTRAN

Actran is software useful for noise analysis. It can study the radiation of sound in external environments and the relation between environment and structures. Actran can predict and analyze the noise radiation even in turbulent and complex flows. It is useful in the study of turbomachinery (MSC Software, 2015).

Actran Aeroacoustics is the simulation tool for complex flow noise. It has direct interface to most leading CFD codes and is compatible with other Actran modules. This tool has efficient solvers, which works in a direct and iterative way, in order to reduce CPU times and loads. Actran Aeroacoustics is useful for the study of air conditioning modules, fan noise or airframe noise (landing gear, trailing edge…). It can be used for the study of the sound in railway axes too. Actran Aeroacoustics works in the following way:

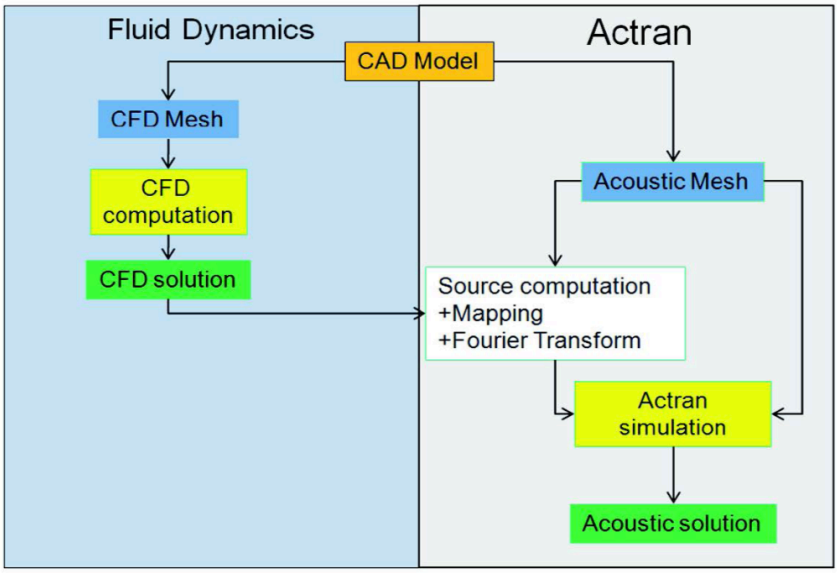

Figure 1 - Actran performance

Briefly, MSC Software is a very powerful group of tools for the analysis of rotating structures and noise radiation. Companies as Boeing use these tools in their studies and analysis. They are easy to use and they give a lot of facilities to the work of the engineers, and they are well linked to other useful software, such as CAD tools, Ansys, Abaqus… In summary, MSC software is one of the most useful and complete commercial codes that exist for the structural analysis of rotating structures.

2.1.4. LMS Virtual.Lab

LMS Virtual.Lab (Siemens, 2006) is another integrated suite of 3D Finite Elements and multibody simulation software, developed by Siemens. It can simulate and analyze the behavior of mechanical systems, studying their structural integrity, their life and other parameters. It is also useful for analyzing vibration and noise radiation of the structures. Using LMS Virtual.Lab it is possible to simulate mechanical design behavior, explore multiple design alternatives, detect weak spots of the structures and optimize designs before prototype construction.

LMS Virtual.Lab is a suite of tools similar to MSC Software, but there are some differences between them. LMS Virtual.Lab is also divided in several packages, and each one is specialized in a particular field.

2.1.4.1. LMS VIRTUAL.LAB NOISE AND VIBRATION

LMS Virtual.Lab Noise and Vibration (Siemens, 2006) combines Finite Element (FE) modeling and test-derived models. It can analyze and predict the forced response and the level of noise of a structure. Noise and Vibration uses data exported from the LMS Virtual.Lab Multibody Modeling Software, which simulates the behavior of multibody and flexible body structures. Noise and Vibration creates data that are useful for LMS Virtual.Lab Acoustics, in order to make acoustic simulations. Using the tools provided by Noise and Vibration, it is possible to create system-level noise, vibration and harshness (NVH) models. These models are based on subsystems and components, and it is possible to modify FE individually or test-derived components with the aim of adapting the software to the studied problem.

The main advantage of LMS Virtual.Lab Noise and Vibration is the fact that it allows the engineer to incorporate models based on FE and models based on tests in a unique assembly. This is a powerful technical feature, because it means that Noise and Vibration provides accuracy and design flexibility. It is important in order to get reliable results.

2.1.4.2. LMS VIRTUAL.LAB ACOUSTICS

LMS Virtual.Lab Acoustics (Siemens, 2006) is a tool for acoustic simulation. It simulates interior and exterior acoustic radiation, and it is specialized on the sound production of the structures, in order to simulate and create quieter structures. It has been designed to minimize noise and optimize sound quality, and it is useful for creating sound meshes. Using Virtual.Lab Acoustics allows the engineers to predict sound fields in complex problems reducing the time spent on it. These predictions used to be achieved by parametric analysis and expensive techniques; making this through computational simulation is cheaper, easier and faster. Besides, computational analysis gives design flexibility, because it is possible to vary a lot of variables.

LMS Sysnoise Technologies is an additional tool that complements LMS Virtual.Lab Acoustics. Both form the first end-to-end environment for acoustic performance using virtual models. Their solutions can be applied to a huge variety of engineering problems, such as structural noise radiation, flow-induced noise and random acoustic loading. LMS Virtual.Lab Acoustics has some subpackages, as for example LMS Virtual.Lab Finite Element Acoustics or LMS Virtual.Lab Boundary Element Acoustics.

2.1.4.3. LMS VIRTUAL.LAB FINITE ELEMENTS ACOUSTICS

This subpackage uses FEM in order to help the engineers predicting and improving sound and noise performances and problems in structures. It can analyze how acoustic sources affect the structure through noise and vibration radiation.

LMS Virtual.Lab Finite Elements Acoustics can model detailed acoustic domains. It allows the engineer to surround a reduced finite mesh, which means big computational savings. It can perform an acoustic simulation without modeling the whole environment, using Perfectly Matched Layer (PML) and Automatically Matched Layer (AML) methods.

FEM can achieve coupled vibro-acoustic simulations in time and frequency domains. In addition, LMS Virtual.Lab Finite Elements Acoustics incorporates advanced FEM solvers, as for example the iterative Krylov solver and the direct MUMPS solver: these tools reduce the time spent in computations.

In short, this subpackage is useful for accounting for multiple material properties, performing calculations, finding the cause of noise problems quickly and predicting acoustic performances accurately, minimizing design risk and reducing time and money costs.

2.1.4.4. LMS VIRTUAL.LAB BOUNDARY ELEMENT ACOUSTICS

The usefulness of this subpackage is similar to Finite Element Acoustics one. It is used to predict and improve sound and noise performance of a broad range of systems, specially focusing on radiation problems. It allows the engineer to achieve good results quickly and easily without compromising accuracy.

The Boundary Element Method (BEM) reduces complex 3D geometry to 2D surface dimensions, as it has been explained in section “Boundary Element Method”. BEM needs to model the surface areas that are vibrating or scattering. This means that the number of elements (and in fact the size of the matrices for the computations) is rather reduced. Hence, it results in a short simulation time.

Modeling coupled problems is possible by using BEM. Coupled problems are common in lightweight structures and structures in contact with a fluid, which is the case of submarines. It is also useful for analyzing the vibration produced in a train tunnel radiated to the floor and the buildings. BEM is particularly suitable for exterior radiation problems, and this subpackage is well related to other common codes, such as Abaqus, Ansys or Nastran.

LMS Virtual.Lab Boundary Element Acoustics uses a fast multipole BEM solver. This process is useful to solve large BEM problems. A typical BEM solver can address BEM models up to 20.000 nodes, and LMS Virtual.Lab Fast Multipole Boundary Element Acoustics can handle more than 1.000.000 nodes. This is a huge advantage in order to analyze aircrafts, ships, submarines... In addition, problems relating to higher frequencies can be tackled using this process.

Fast Multipole BEM Solver uses iterative techniques with additional and advanced algorithms. These algorithms substructures cells in different multilevel hierarchically, and the model divides the problem into several domains, instead of solving to whole model at the same time. This means that this solver can divide its work in several computers or clusters in parallel. This solver can also study the acoustics of the environment.

Briefly, the main advantages of Fast Multipole BEM Solver are the number of nodes that can be studied, the speed in the solution, the fact that it complements other BEM techniques, the reduction of preprocessing time (less stringent mesh requirements), and the increasing of the frequency range of analysis.

2.1.4.5. LMS VIRTUAL.LAB OPTIMIZATION

It is necessary to validate the models in order to prevent accumulating inaccuracies. To ensure this fact, engineers can compare the models with experimental data or validated models of similar structures. It is important not to have issues on material properties, boundary conditions and other relevant parameters.

LMS Virtual.Lab Correlation (Siemens, 2006) can correlate physical tests results and prepare structural tests easily. LMS Virtual.Lab Correlation identifies specific locations that are improvable. LMS Virtual.Lab can also detect the most influential parameters in the studies.

2.1.4.6. STATISTICAL ENERGY ANALYSIS

This tool can be used as a complement of LMS Virtual.Lab (Siemens, 2006) It is useful for helping and complementing LMS Virtual.Lab on acoustic analysis and noise transmission predictions. Statistical Energy Analysis is useful for solving mid to high frequency problems for full systems such as trains, aircrafts… This software simulates the vibro-acoustic energy flow between subsystems of the full system. Analyzing the noise and vibration transmission it is possible to determine the cause of the problem and to reduce it.

2.1.5. In-house code

MSC Software and LMS Virtual.Lab are powerful and useful sets of tools for structural analysis, noise radiation, vibration and calculus of materials life, among other parameters. However, they have two major problems when using them for the required studies by LEAM.

The first problem is the fact that they are too generic and they are not efficient for the analyses required here, as it has been said in background section. These tools have been developed for using them in a lot of engineering cases, so they are not adequate for some cases. They have not been optimized for a specific case, because they have to be useful for a lot of cases. Accordingly, MSC Software and LMS Virtual.Lab are useful depending on the problem.

The second problem is the fact than MSC Software cannot work in BEM and Virtual.Lab cannot deal with rotating structures. Therefore, Virtual.Lab cannot be used for the problem presented in this project, and MSC Software have not the computational savings that BEM represents for noise radiation problems. Another problem is the price: using these commercial codes is not cheap, and worth developing a specific, optimized and efficient new code if only very specific calculations want to be performed.

As it has been explained before, the part of the code that is going to be developed will be based on an existing code made by LEAM members (de Haro & Arcos, 2014), which calculates the vibration response of non-rotating structures. This code is based on FEM, using first order hexahedral finite elements. It builds all the matrices for each element, such as the stiffness matrix, the mass matrix, the force vector… The process of obtaining these matrices is based on FEM shape functions, integrating them analytically (the integrals can also be done numerically, but the analytic solution is exact). After that, it assembles the elemental matrices in a global matrix. Finally, it reduces the system by eliminating the rows and columns of the matrices that corresponds to restricted degrees of freedom (DOF). The information of the nodes is given by GiD, the mesh creator, and processed by MATLAB. It is important to notice that a GiD “problem type” had been created: this fact simplifies the work, because it writes data in a specific way, adequate for the LEAM’s in-house MATLAB code.

2.2. Proposed solution

Due to the problems of the existing commercial codes, the proposed solution is the development of an in-house code. The idea of modifying the existing commercial codes in order to adapt them to the requirements of LEAM problems is not an option, because transforming them is not possible because the companies do not provide the source codes. In spite of this, modifying the existing in-house code is possible.

In order to develop a new code, the continuation of the existing one, the study and comprehension of the dynamics of rotating structures is essential.

In spite of this, even though the vibration response of rotating structures has been studied less than the vibration response of non-rotating ones, there are some publications that analyze such rotating problems. One of the most important publications in the computational study of rotating structures is the book that had been written by researchers Arne Vollan and Louis Komzsik, and it is called “Computational Techniques of Rotor Dynamics with the Finite Element Method” (Vollan & Komzsik, 2012). Techniques proposed by Arne Vollan and Louis Komzsik are very useful. Both used to work for NASTRAN, which is the one of the leading FEM analysis tools in structural engineering. Techniques devised by Vollan and Komzsik allow engineers to make their own code. They use the nodal notation for all their analysis, which means that all the matrices are computed for each node, not for each element, but their formulation can be adapted to the formulation of the in-house code.

2.3. Theoretical foundation

2.3.1. Introduction

A. Vollan and L. Komzsik have developed the theoretical foundation of computational techniques for the study of rotor dynamics using FEM. Understanding rotor dynamics is basic in order to develop computational codes for the analysis. After understanding rotor dynamics, which includes a good knowing of fixed and rotating coordinate systems, the forces, the transformations between systems, the kinetic energy and the equations of motion, A. Vollan and L. Komzsik study coupled solution formulations. This is necessary if the structure is rotating and translating at the same time and if each node has translational and rotational DOF (this topic is explained afterwards in this section). The book is very complete and allows the reader to understand and develop codes based on the fixed or the coordinate system with translation or rotation (or both) and to analyze the non-rotating and rotating parts of a structure separately.

The following topic achieved on the book is FEM analysis of rotating structures. All the equations must be written in the matrix form, which relates the nodes between themselves. Basically, the equations are the same for rotating structures than for non-rotating structures, but the rotating ones have some additional components, obviously due to the rotation. These additional components should be computed and added to the existing code.

In FEM, the degrees of freedom (DOF from now) of each node can be selected depending on the case. Considering that each node has only 3 DOF (the translation in each axis) is common in complex 3D structures, because normally the angles between the original mesh and the deformed one are small, and the orientation of the nodes has not a big importance (only the displacements are important). However, if the structure is similar to a beam, rotations have to be computed as DOF. The reason is the fact that the deformation of a beam cannot be explained if rotations are not computed. The in-house code only uses 3 DOF: these DOF are the translations of the nodes in each axis.

2.3.2. Finite Element Analysis of rotating structures

A structure can be discretized using Finite Elements (FE), transforming a continuous system to a NDOF system, which is a system with N number of degrees of freedom. The assembly of the motion equations of all the FE gives the equilibrium equations of the nodes of the whole system. In order to get the equilibrium equations, it is necessary to write the position, velocity and acceleration in generalized coordinates. Matrix $\{g\}$ contains the matrices of generalized coordinates of all the nodes, which are $\{g_i\}$. The subscript i goes from 1 to N, which is the number of nodes. The first 3 DOF in $\{g_i\}$ are the translations, and the last ones are the rotations.

$$\{g\} = \begin{Bmatrix} \{g_1\} \\ \{g_2\} \\ . \\ . \\ . \\ \{g_i\} \\ . \\ . \\ . \\ \{g_n\} \end{Bmatrix}, \qquad \{g_i\} = \begin{Bmatrix} u_i \\ v_i \\ w_i \\ \varphi_i \\ \psi_i \\ \theta_i \end{Bmatrix} \tag{2.1}$$

The equilibrium equation comes from Lagrange equation. Lagrange equation for a whole system is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\{\dot{g}_i\}}\right) - \frac{\partial L}{\partial\{g_i\}} + \frac{\partial D}{\partial\{\dot{g}_i\}} = \frac{dW}{\partial\{g_i\}} \quad ; \quad i = 1,2,\dots,n\,, \tag{2.2}$$

where D is the dissipative energy and W is the work of non-dissipative forces. Nondissipative forces are the ones that come from a potential. The Lagrange potential, also called Lagrangian, is the difference between kinetic (T) and potential energy (U) of a particle:

$$L = T - U \tag{2.3}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\{\dot{g}_i\}}\right) - \frac{\partial T}{\partial\{g_i\}} + \frac{\partial U}{\partial\{g_i\}} + \frac{\partial D}{\partial\{\dot{g}_i\}} = \frac{dW}{\partial\{g_i\}} \quad ; \quad i = 1,2,\dots,n \tag{2.4}$$

2.3.2.1. POTENTIAL ENERGY OF STRUCTURE

A rotatory particle is part of a system subjected to elastic deformations. The potential energy due to elastic deformations is:

$$U = \frac{1}{2}\int (\{\sigma\}_0 + \{\sigma\})^T \cdot (\{\varepsilon\} + \{\varepsilon\}_l)\, dV \tag{2.5}$$

$$U = \frac{1}{2}\int (\{\sigma\}_0^T\{\varepsilon\} + \{\sigma\}_0^T\{\varepsilon\}_l + \{\sigma\}^T\{\varepsilon\} + \{\sigma\}^T\{\varepsilon\}_l)\, dV \tag{2.6}$$

$$\{\sigma\}_0 : initial\ stress \quad \{\varepsilon\}_l : large\ strains$$

$\{\sigma\}_0{}^T\{\varepsilon\}$ and $\{\sigma\}^T\{\varepsilon\}_l$ are neglected because they are second order terms. Then,

$$U = \frac{1}{2}\int \{\sigma\}_0^T\{\varepsilon\}_l\, dV + \frac{1}{2}\int \{\sigma\}^T\{\varepsilon\}\, dV \tag{2.7}$$

The first term is the potential energy due to initial stress U_G, and the second one is the elastic potential energy U_E.

- Elastic potential energy U_E

The relation between strains and stresses is linear: a material matrix $[E]$ containing Young's modulus and Poisson ratio relates them in the following way:

$$\{\sigma\} = [E]\{\varepsilon\} \tag{2.8}$$

where

$$\{\sigma\} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} \quad ; \quad \{\varepsilon\} = \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial w/\partial z \\ \partial u/\partial y + \partial v/\partial x \\ \partial u/\partial z + \partial w/\partial x \\ \partial v/\partial z + \partial w/\partial y \end{Bmatrix} \tag{2.9}$$

The linear strain is a function of nodal generalized displacements:

$$\{\varepsilon\} = [B]\{g\} \tag{2.10}$$

In this expression $\{\varepsilon\}$ is the linear strain, $[B]$ is a matrix containing the shape functions (which depend on the type of elements) and $\{g\}$ is the matrix containing the generalized displacements. Knowing this, it is possible to write the expression of elastic potential energy:

$$U_E = \frac{1}{2}\int \{\sigma\}^T\{\varepsilon\}\ dV = \frac{1}{2}\int \{g\}^T\,[B]^T[E][B]\{g\}\,dV \tag{2.11}$$

$$\frac{dU_E}{d\{g\}} = \left(\int [B]^T[E][B]\,dV\right)\{g\} = [K]\{g\} \tag{2.12}$$

Matrix $[K]$ is called the elastic stiffness matrix.

- Potential energy due to initial stress U_G

This term contains the large strain vector, which is:

$$\{\varepsilon\}_l = \begin{Bmatrix} \{\rho\}_x^T\{\rho\}_x \\ \{\rho\}_y^T\{\rho\}_y \\ \{\rho\}_z^T\{\rho\}_z \\ \{\rho\}_x^T\{\rho\}_y + \{\rho\}_y^T\{\rho\}_x \\ \{\rho\}_x^T\{\rho\}_z + \{\rho\}_z^T\{\rho\}_x \\ \{\rho\}_y^T\{\rho\}_z + \{\rho\}_z^T\{\rho\}_y \end{Bmatrix} \tag{2.13}$$

The subscripts means

$$\{\rho\}_* = \frac{\partial\{\rho\}}{\partial *} \tag{2.14}$$

And $\{\rho\}$ is a vector containing the translations in the three axes of any point of the structure:

$$\{\rho\} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} \tag{2.15}$$

After a mathematical process and knowing that $\sigma_{**}\{\rho\}_*^T\{\rho\}_* = \{\rho\}_*^T\sigma_{**}\{\rho\}_*$, the expression of potential energy due to initial stress can be written in the following way:

$$U_G = \frac{1}{2}\int \{\sigma\}_0^T\{\varepsilon\}_l \, dV = \frac{1}{2}\int \{e\}_l^T[S]_0\{e\}_l \, dV \tag{2.16}$$

In this last expression, $\{e\}_l$ is a vector related to the displacements through the derivatives of the shape functions. Thus, it can be written similarly to $\{\varepsilon\}$:

$$\{e\}_l = [B']\{g\} \tag{2.17}$$

where $[B']$ is a matrix containing the derivatives of the shape functions. The new expression of U_G contains another new term, $[S]_0$. This matrix must contain the initial stress terms, because they cannot disappear. The form of this matrix is

$$[S]_0 = \begin{bmatrix} \sigma_{xx}I_3 & \sigma_{xy}I_3 & \sigma_{xz}I_3 \\ \sigma_{yx}I_3 & \sigma_{yy}I_3 & \sigma_{yz}I_3 \\ \sigma_{zx}I_3 & \sigma_{zy}I_3 & \sigma_{zz}I_3 \end{bmatrix} \tag{2.18}$$

where I_3 is the identity matrix with 3 rows and columns. It is important to notice that initial stress is always the result of external forces, such as the centrifugal force. These external forces depend on the rotation speed. Therefore, the expression of the derivative of potential energy due to initial stress yields:

$$\frac{dU_G}{d\{g\}} = \left(\int [B']^T[S]_0[B'] \, dV\right) \cdot \{g\} = \Omega^2[K_G]\{g\} \tag{2.19}$$

Matrix $[K_G]$ is called the geometrical or differential stiffness matrix. It is important to notice that the size of both stiffness matrices ($[K_G]$ and $[K]$) is the same. These matrices are square matrices, and their size is the total number of nodes multiplied by the number of local DOF.

2.3.2.2. DISSIPATIVE FORCES

The dissipative term on Lagrange equation represents the energy lost in the structure. In a first approach, it is common considering them null. The dissipative forces damp the kinetic energy of the particles, and they form the damping matrices. Those forces can be external or internal.

- Internal dissipative forces

The damping between particles of the structure produces the internal dissipative forces. The Rayleigh function explains this phenomenon (Kilminster, 1970). The Rayleigh dissipation function is defined through dissipation coefficients, which depends on the material properties and the geometry. The coefficients $d_{i,j}$ and the generalized velocity $\{\dot{g}\}$ of each particle show how a particle i affects another particle j.

$$D_f = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} d_{i,j} \cdot \{\dot{g}_i\}^T\{\dot{g}_j\} \tag{2.20}$$

Translations and rotations of the particles create damping, and the effect of each motion is represented in a matrix. The one related to the translations is $[D_\rho]_i$, and the related to the rotations is $[D_\alpha]_i$. The subscript i shows that there is a matrix for each node. The augmentation of the two matrices generates the matrix that represents the internal damping forces for each node $[D_I]_i$. The form of this matrix is:

$$[D_I]_i = \begin{bmatrix} \mathrm{d}_{i,u} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathrm{d}_{i,v} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{d}_{i,w} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{d}_{i,\varphi} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{d}_{i,\psi} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathrm{d}_{i,\vartheta} \end{bmatrix} \tag{2.21}$$

- External dissipative forces

These forces are usually originated in the bearing, and they carry the effect of the bearing to the rotating structure, mainly in a plane perpendicular to the axis of rotation. These forces are proportional to the nodal velocity. After some mathematical processes, the bearing induced damping force in the rotating reference system can be defined as:

$$\{F\}_i = -\Omega d_{F,i}[P]^T\{\rho\}_i - d_{F,i}[I]\{\dot{\rho}\}_i \tag{2.22}$$

$$[P] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.23}$$

The coefficients $d_{F,i}$ are caused by the external forces. The first term of the previous equation is the circulatory matrix, and it is the damping component associated with the stiffness term.

$$d_{F,i} \cdot [P]^T = \begin{bmatrix} 0 & d_{F,i} & 0 \\ -d_{F,i} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = [K_D]_{\rho,i} \tag{2.24}$$

The second term is simply the damping matrix:

$$d_{F,i} \cdot [I] = \begin{bmatrix} d_{F,i} & 0 & 0 \\ 0 & d_{F,i} & 0 \\ 0 & 0 & d_{F,i} \end{bmatrix} = [D_E]_{\rho,i} \tag{2.25}$$

The same process can be done with the damping created by moments. In this case, matrices are:

$$[K_D]_{\alpha,i} = \begin{bmatrix} 0 & d_{M,i} & 0 \\ -d_{M,i} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.26}$$

$$[D_E]_{\alpha,i} = \begin{bmatrix} d_{M,i} & 0 & 0 \\ 0 & d_{M,i} & 0 \\ 0 & 0 & d_{M,i} \end{bmatrix} \tag{2.27}$$

Finally, it is possible to join force and moment matrices. The external circulatory matrix has the following form:

$$[D_E]_i = \begin{bmatrix} 0 & \mathrm{d}_{F,i} & 0 & & & \\ -\mathrm{d}_{F,i} & 0 & 0 & & & \\ 0 & 0 & 0 & & & \\ & & & 0 & \mathrm{d}_{M,i} & 0 \\ & & & -\mathrm{d}_{M,i} & 0 & 0 \\ & & & 0 & 0 & 0 \end{bmatrix} \tag{2.28}$$

And the external damping matrix has the following form:

$$[D_E]_i = \begin{bmatrix} \mathrm{d}_{F,i} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathrm{d}_{F,i} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{d}_{F,i} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{d}_{M,i} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{d}_{M,i} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathrm{d}_{M,i} \end{bmatrix} \tag{2.29}$$

The effect of internal and external dissipative forces can be showed together. The matrices related to the velocity are usually added:

$$[D]_i = [D_E]_i + [D_I]_i \tag{2.30}$$

It can be done because their location and multiplier term in the equilibrium equation is identical. This new matrix $[D]_i$ represents the viscous damping of the system (not only caused by viscous dampers).

2.3.2.3. NONDISSIPATIVE FORCES

These forces are of two kinds: the conservative forces acting on the structure constitute the first kind, and the second kind arises from an active external physical phenomenon interacting with the rotating structure.

- Conservative forces acting on the structure

Conservatives forces are the ones that derive from a vector potential. It means that their work do not depend on the path taken by the body on a closed curve in the vector field. A clear example of a conservative force is the gravity. These forces usually act throughout the whole body of the structure, and therefore:

$$W_p = -\iiint \{f_p\}\{g\}\, dV \tag{2.31}$$

$$\{F_p\} = -\frac{dW_p}{d\{g\}} \tag{2.32}$$

Where $\{F_p\}$ is the global vector containing the conservatives forces that acts on the volume of the structure.

- Active external physical phenomenon interacting with the rotating structure

Structures that are not located in the vacuum are subjected to the interaction with the fluid field that surrounds them. The work made by the fluid on the structure is defined by a surface integral:

$$W_a = -\iint \{f_a\}\{g\}\, dS \tag{2.33}$$

$$\{F_a\} = -\frac{dW_a}{d\{g\}} \tag{2.34}$$

Where $\{F_a\}$ is the global vector that contains all active forces acting on the surface of the structure.

- Centrifugal force

The centrifugal force $\{F_c\}$ is also a volume force (the conservative forces are volume forces), so it has to be added to the total sum of nondissipative forces. Finally, the nondissipative forces results in the following expression, which occupies the right-hand side of the equilibrium equation:

$$\{F\} = \{F_c\} + \{F_V\} + \{F_a\} \tag{2.35}$$

2.3.2.4. FINITE ELEMENT EQUATION ASSEMBLY

The equilibrium equation for a rotatory structure in a rotatory reference system is:

$$[M]\{\ddot{g}\} + ([D] + 2\Omega[C])\{\dot{g}\} + ([K] - \Omega^2[Z] + \Omega^2[K_G] + \Omega[K_D])\{g\} = \{f\} \tag{2.36}$$

And the equilibrium equation for a rotatory structure in a fixed reference system is:

$$[M]\{\ddot{g}\} + ([D] + \Omega[C])\{\dot{g}\} + ([K] + \Omega[K_D])\{g\} = \{f\} \tag{2.37}$$

$[M]$ is the mass matrix. $[D]$ is the damping matrix, and it can be modeled following the Rayleigh equation. Its definition comes from coefficients d, but knowing the exact value of these coefficients is difficult, so modeling the damping using some approximations is common. Hence, damping matrix can be (Arcos, 2015)

$$[D] = \alpha[M] + \beta[K] \tag{2.39}$$

where α and β are arbitrary constants. This kind of damping is called proportional damping. It is usual to consider the proportional damping null, as a first approach.

Terms containing Ω or $[C]$ are added to the equation due to the rotation of the structure. The term Ω is the rotational speed of the structure (this rotational speed is not the same as the ω calculated through the eigenvalues). $[C]$ is the gyroscopic matrix. $[K]$ is the elastic stiffness matrix, which depends on the material properties and the geometry, and $[K_D]$ is the damping component associated with the stiffness term (it depends directly on the damping matrix, so it is null in the in-house code). $[K_G]$ is called the differential or geometric stiffness matrix, and its definition is written in section 2.3.2.1. This matrix is caused by the initial stress. If there is not initial stress, this matrix is null. Matrix $[Z]$ is called the centrifugal softening matrix, and its definition is similar to the definition of the gyroscopic matrix. The right-hand side of the equations is occupied by $\{f\}$, the vector of forces.

2.3.2.5. GYROSCOPIC AND CENTRIFUGAL SOFTENING MATRIX CONSTRUCTION

Now, the equation form is known. The main issue is the construction of $[C]$ and $[Z]$. As it has been explained before, the nodes in the in-house code have only 3 DOF, because in the 3D structures that are going to be studied it is not necessary to consider rotations of the nodes. Therefore, only the modifications caused by translations (which are indicated by the subscript ρ) of the nodes must be added. This fact simplifies the work. Hence, $[C]$ is formed only by the translation terms. This is called the Coriolis term of the gyroscopic matrix. According to Vollan and Komszik:

$$[C_\rho] = \begin{bmatrix} 0 & -m & 0 \\ m & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.40}$$

The centrifugal softening matrix is:

$$[Z_\rho] = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.41}$$

And the mass matrix is:

$$[M_\rho] = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \tag{2.42}$$

Relating the matrices is easy, as it can be seen observing their composition carefully. The first column of $[C_\rho]$ is equal to the second one of $[M_\rho]$, and the second column of $[C_\rho]$ is equal to the first one of $[M_\rho]$ changing the sign. The construction of $[Z_\rho]$ consists on eliminating the last column of $[M_\rho]$. The mass matrix is data of the problem, so the gyroscopic matrix can be built using it. It is important to note that these matrices belong to each node, and each row and column corresponds to a DOF (each node has 3 DOF). Hence, the global $[C]$ matrix can be built using $[M]$, which is already computed by the existing code.

2.3.3. Equation analysis

As it could be expected, there are some differences between the expression in a fixed reference system and the expression in a rotating reference system. The equation in a fixed system does not have the centrifugal and the differential stiffness matrices. There are also some changes in the coefficients that appear before some matrices. The Coriolis terms are missing in the fixed system, because this force is only detectable in the rotating system.

The most interesting equation for the in-house study is the second one, which is the equilibrium equation for a rotatory structure in a fixed system. Knowing the eigenvalues from an external point of view is important because the study is centered on calculating the external effects of the rotation (the main aim of the whole project is the analysis of noise radiation, so the most interesting point of view is the external one). In order to solve the problem, modal method is more useful than direct method in our case, but there are other engineering or physical problems that need the direct method. The direct method solves the whole equation system, and it is useful computationally only if the number of nodes is small. The modal method uses the eigenvalues to transform the system and to solve it.

2.3.4. Equation development

2.3.4.1. FIXED REFERENCE SYSTEM

The general equation in a fixed reference system is the following:

$$[M]\{\ddot{g}\} + ([D] + \Omega[C])\{\dot{g}\} + ([K] + \Omega[K_D])\{g\} = \{f\} \tag{2.43}$$

As terms related to damping are considered null, the equation yields:

$$[M]\{\ddot{g}\} + \Omega[C]\{\dot{g}\} + [K]\{g\} = \{f\} \tag{2.44}$$

This equation is not exactly the same as the equation for fixed structures without damping. The existing in-house code works with this kind of structures, but there are some differences between the new equation and the equation for fixed structures. These differences can vary the process of solving the problem. The equation for fixed structures without damping in time domain is:

$$[M]\{\ddot{g}\} + [K]\{g\} = \{f\} \tag{2.45}$$

It is easy to realize that some new terms have appeared in the rotatory equation: the rotational speed and the gyroscopic matrix. Knowing the internal composition of $[C]$ is important, and also studying how these new terms affect the resolution of the equation system. Before studying these changes, it is necessary to know that if the aim is computing the eigenvalues, the code must work in frequency domain. The equation for fixed structures without damping can be transformed to frequency domain through the transference function:

$$(-\omega^2[M] + [K]) \cdot \{G\} = \{F\} \tag{2.46}$$

In order to compute the eigenvalues, $\{F\}$ is considered 0 because free response is searched. Eigenvalues can be found solving this equation:

$$det(-\lambda[M] + [K]) = 0 \rightarrow \lambda = \omega^2 \tag{2.47}$$

Although this equation is not the same as the equation for rotatory structures, the equation for non-rotating structures with damping is really similar to the rotating one. The equation for a fixed structure with damping in time domain is:

$$[M]\{\ddot{g}\} + [D]\{\dot{g}\} + [K]\{g\} = \{f\} \tag{2.48}$$

Notice that there is only one difference between the equation for a fixed structure with damping and the equation for a rotatory structure: term $[D]$ is substituted by $\Omega[C]$. Transforming the equation for fixed structures with damping to frequency domain is possible:

$$(-\omega^2[M] + i\omega[D] + [K]) \cdot \{G\} = \{F\} \tag{2.49}$$

The state-space formulation is used in fixed structures with damping, but it can also be used in our case, because both have similar forms, as we have seen. The state-space formulation in a fixed structure with damping is the following (neglecting forces):

$$\begin{Bmatrix}\{\dot{g}\}\\ \{\ddot{g}\}\end{Bmatrix} - \begin{bmatrix}0 & [I]\\ -[M]^{-1}[K] & -[M]^{-1}[D]\end{bmatrix}\begin{Bmatrix}\{g\}\\ \{\dot{g}\}\end{Bmatrix} = 0 \tag{2.50}$$

Thus, the equation of a rotatory structure can be written using the state-space formulation:

$$\begin{Bmatrix}\{\dot{g}\}\\ \{\ddot{g}\}\end{Bmatrix} - \begin{bmatrix}0 & [I]\\ -[M]^{-1}[K] & -[M]^{-1}\Omega[C]\end{bmatrix}\begin{Bmatrix}\{g\}\\ \{\dot{g}\}\end{Bmatrix} = 0 \tag{2.51}$$

It can be written in compact form:

$$\{\dot{u}\} - [A]\{u\} = 0 \tag{2.52}$$

where $[A]$ is the state matrix and $\{u\}$ is the state vector. This equation can be transformed to frequency domain using the transference function (Arcos, 2015):

$$i\omega\{U\} - [A]\{U\} = 0 \tag{2.53}$$

And now, the eigenvalue problem is:

$$(\lambda[I] - [A]) = 0 \tag{2.54}$$

In this case λ is not equal to ω^2 (this happens in non-rotatory problems): it is equal to $i\omega$.

2.3.4.2. ROTATING REFERENCE SYSTEM

Studying the eigenvalues in the rotating reference system is useful, in order to discover the internal effects of the rotation in the structures. The equation in the rotating reference system is not the same as the one in the fixed reference system. Then, the equation should be:

$$[M]\{\ddot{g}\} + ([D] + 2\Omega[C])\{\dot{g}\} + ([K] - \Omega^2[Z] + \Omega^2[K_G] + \Omega[K_D])\{g\} = \{f\} \quad (2.55)$$

Terms $[D]$ and $[K_D]$ are considered null, as before. $[K_G]$ is also considered null because there will not be initial stress in the structures analyzed through the code. Then, the equation yields:

$$[M]\{\ddot{g}\} + (2\Omega[C])\{\dot{g}\} + ([K] - \Omega^2[Z])\{g\} = \{f\} \quad (2.56)$$

2.3.4.3. FORMULATION OPTIMIZATION

The state-space formulation is useful for this kind of analysis, but it has some computational problems. The operation of inverting a matrix takes a lot of time, and it could need more RAM than the RAM that has the computer. Then, this state-space formulation should be modified in order to avoid the inverse of a matrix. It can be done multiplying the equation by $[M]$. This new state-space formulation is valid for both fixed and rotating reference systems.

$$fixed\ system: \begin{bmatrix}[M] & 0\\ 0 & [M]\end{bmatrix}\begin{Bmatrix}\{\dot{g}\}\\ \{\ddot{g}\}\end{Bmatrix} - \begin{bmatrix}0 & [M]\\ -[K] & -\Omega[C]\end{bmatrix}\begin{Bmatrix}\{g\}\\ \{\dot{g}\}\end{Bmatrix} = 0 \quad (2.57)$$

$$rotating\ system: \begin{bmatrix}[M] & 0\\ 0 & [M]\end{bmatrix}\begin{Bmatrix}\{\dot{g}\}\\ \{\ddot{g}\}\end{Bmatrix} - \begin{bmatrix}0 & [M]\\ -([K] - \Omega^2[Z]) & -2\Omega[C]\end{bmatrix}\begin{Bmatrix}\{g\}\\ \{\dot{g}\}\end{Bmatrix} = 0 \quad (2.58)$$

$$[M']\{\dot{u}\} - [A']\{u\} = 0 \quad (2.59)$$

$$i\omega[M']\{U\} - [A']\{U\} = 0 \quad (2.60)$$

And now, the eigenvalue problem is:

$$(\lambda[M'] - [A']) = 0 \quad (2.61)$$

$$\lambda = i\omega \quad (2.62)$$

This formulation avoids processes of inverting matrices, so it is more efficient computationally.

2.3.5. Development of the chosen solution

After learning about the theoretical base, it is time to transform the existing MATLAB code. As it has been explained, damping terms are considered null as a first approach. Hence, the only terms that have to be added to the existing equation is the gyroscopic matrix and the centrifugal softening matrix. Using MATLAB, working with the global mass matrix is useful, because its structure is simple. The code changes the corresponding columns between themselves in order to construct the new matrices. It is important to notice that the code have to select the first and second columns of each group of 3 columns, because the structure is repeated every 3 columns (every 3 columns corresponds to a node, which have 3 DOF).

It is also important to know that the code works with the global matrix without the columns and rows that correspond to restricted DOF (nodes with non-movement). Constructing the new matrices in this way is a good option: if the chosen process were the construction of the local gyroscopic matrix for each node, another assembling process would have to be added. An example in a simple structure with 2 nodes is attached. All the information of the nodes and elements is given by a text file, which comes from GiD. This information and the theoretical base of FEM provide the stiffness and mass matrices, and the process of their creation is already done by the existing code.

$$[M] = \begin{bmatrix} 2 & 0 & 0 & 7 & 0 & 0 \\ 0 & 2 & 0 & 0 & 7 & 0 \\ 0 & 0 & 2 & 0 & 0 & 7 \\ 3 & 0 & 0 & 4 & 0 & 0 \\ 0 & 3 & 0 & 0 & 4 & 0 \\ 0 & 0 & 3 & 0 & 0 & 4 \end{bmatrix}$$

$$[C] = \begin{bmatrix} 0 & -2 & 0 & 0 & -7 & 0 \\ 2 & 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & -4 & 0 \\ 3 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[Z] = \begin{bmatrix} 2 & 0 & 0 & 7 & 0 & 0 \\ 0 & 2 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 4 & 0 & 0 \\ 0 & 3 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The process of obtaining the information of the nodes and elements is the following: firstly, uploading the "problem type" in GiD is essential. This is useful because this "problem type" makes GiD work in a concrete way, giving the information in a specific form. Secondly, a figure has to be created in GiD, and after this the geometry has to be meshed with hexahedral 3D elements. Once the mesh is created, the only step left is extracting the text file with the required information of the nodes and elements.

Once the gyroscopic matrix is built, it has to be added to another new matrix: the state matrix. This matrix is composed by the mass matrix, a zero matrix, an identity matrix, the rotational speed and the gyroscopic matrix (eq. 2.57). This state matrix is, as it can be expected, of the size 2NDOF·2NDOF (remember that now NDOF is the total number of nodes multiplied by 3, which are the DOF of each node, minus the number of restricted nodes multiplied by 3), so 2NDOF eigenvalues can be found. The user can choose the rotational speed, and the axis of rotation is z-axis. The construction of the structure must take into account this feature of the code. The reason why this matrix is a square matrix 2NDOF·2NDOF is its formulation: the first NDOF columns and rows correspond to position NDOF, and the second ones correspond to velocity NDOF. The state-space formulation (eq. 2.57) works with the vector $\{u\}$, which has the position coordinates and the velocity coordinates.

Obtaining the eigenvalues and eigenvectors is easy by using MATLAB, because it has a mathematical optimized function, called "eigs", which provides a matrix with the eigenvalues on its diagonal and another matrix with the corresponding eigenvectors of a given matrix (in this case, the state matrix).

Once the eigenvalues and eigenvectors are calculated, they must be analyzed and studied. This study is done in section 3.2. onwards. Furthermore, this code can be modified in the future in order to achieve more complex steps, with the final aim of the study of the noise radiation of rotating structures.

2.3.5.1. CODE OPTIMIZATION

It is important to notice that the structures that are going to be studied would have a lot of nodes. There should be enough nodes to get reliable results.

This fact creates a problem in the code, because MATLAB is not able to work with matrices so large (each node has 3 DOF, so as the number of nodes increase, the size of the matrices increase too). Hence, the code must be modified again. All the matrices have to be of the kind "sparse". This kind of matrix occupies much less memory than a normal matrix, which can be called "full". "Sparse" only stores the positions of the matrix where the value is different to zero and these values, so "sparse" represents big memory savings. Instead of storing over $(3\cdot NDOF)^2$ number of values of the kind "double", these "sparse" matrices store much less values of the kind "double", because the stiffness matrix, the mass matrix and the gyroscopic matrix have a lot of zeros in their composition.

3. Results

The main result is the code itself, but it is interesting to use it in some cases in order to check its usefulness. The study of the results requires a previous knowledge about complex solutions, in order to analyze it in a correct way.

3.1. Complex solution

The fact that term $\Omega[C]$ appears in the new equation multiplying the velocity implies that the solution can be not pure real. Hence, a rotatory structure has 2·NDOF eigenvalues, and some of them are complex ones (there can be also real eigenvalues). As eigenvalues are complex, the corresponding natural frequencies are complex, unless the eigenvalue is a pure complex number, because in the studied structures the eigenvalues are $\lambda = i\omega$. Thus, pure real eigenvalues correspond to pure complex natural frequencies due to the last expression.

3.1.1. Complex eigenvalues

Complex eigenvalues are usually studied in fixed systems with damping, and they represent underdamped motions. They also appear in rotating structures, as it is explained below. If there exists a complex eigenvalue, there always exists its conjugate, which has the same real part and the same imaginary part with the sign changed.

The modulus of the whole eigenvalue represents the natural frequency ω_n, and the modulus of the imaginary part represents the oscillation frequency ω_d. Function Im represents the imaginary part of a complex number, and Re represents the real part of a complex number.

$$\omega_n = |\lambda| = \sqrt{\mathrm{Im}(\lambda)^2 + \mathrm{Re}(\lambda)^2} \tag{3.1}$$

$$\omega_d = |\mathrm{Im}(\lambda)| \tag{3.2}$$

This means that the real part represents the reduction of the amplitude, and the imaginary part represents the oscillation. It can be understood observing the equation of a damped movement.

$$x = Ae^{-i\omega t} = Ae^{-\lambda t} = A\left(e^{-Re(\lambda)\cdot t} \cdot e^{-i\cdot Im(\lambda)\cdot t}\right) \tag{3.3}$$

The first term is a negative exponential function, which causes the reduction of the amplitude, as it has been previously supposed. The second term is an imaginary exponential function, and it can be developed using Euler's formula (Weisstein, 2015).

$$e^{ix} = \cos x + i \sin x \tag{3.4}$$

$$e^{-i\cdot \mathrm{Im}(\lambda)\cdot t} = -[\cos(\mathrm{Im}(\lambda)\cdot t) + i \sin(\mathrm{Im}(\lambda)\cdot t)] \tag{3.5}$$

Hence, the sine and cosine express the oscillation movement. The following generic graph shows the combination of both terms. The blue line is the decay of the amplitude, and the red line is the oscillation.

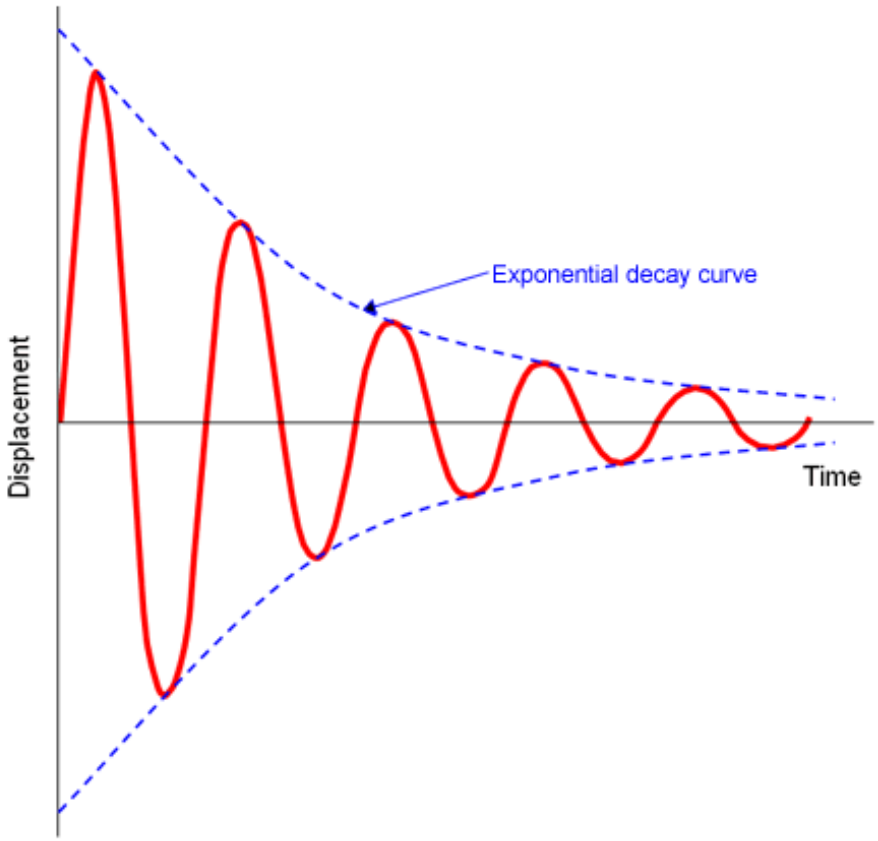

Figure 2 - Damped oscillation (Patana, 2014)

3.1.1.1. RELATION BETWEEN DAMPED NON-ROTATING STRUCTURES AND ROTATING STRUCTURES

There is a parallelism between fixed structures with damped and rotating structures, as it has been expected. The rotation generates a sort of damping in the structure, but there is a main difference between the "rotation damping" and the damping. The motion equation of rotating and damped structures is similar, because both have a matrix multiplying the velocity. The difference comes from the internal composition of these matrices. The damping matrix, as it has been explained before, is a lineal combination between inertia and stiffness matrix: therefore, it is always a symmetric matrix. On the contrary, the gyroscopic matrix is always skew-symmetric, so the system response cannot be the same in a fixed damped system and in a rotating system.

3.2. 3D Cube. Non-rotating and rotating results

As it has been explained in this report, the main goal of this project is the analysis of the effect of the rotation in the vibration response in 3D structures. In order to achieve this objective, comparing the eigenvalues and eigenvectors obtained in a non-rotating structure and the eigenvalues and eigenvectors in the same structure with rotation is useful. Furthermore, comparing the eigenvalues with different velocities of rotation is interesting. The chosen structure is a cube made by steel with dimensions equal to 1 meter, because the previous code used this structure and the eigenvalues are correct.

Density	$\rho = 7850\ \mathrm{kg/m^3}$
Young's modulus	$E = 2.07 \cdot 10^{11}\ \mathrm{Pa}$
Poisson ratio	$\nu = 0.40$
Length, width and height	$c = 1\ \mathrm{m}$

Table 1 - Cube material properties

3.2.1. 3D Cube. Non-rotating case

The study of the non-rotating case can be carried out using the existing code, because there is not rotation, and the calculation of the eigenvalues can be made through equation 2.47. Thus, the obtained results for a non-rotating cube are:

Mode number	**Eigenvalue λ [rad/s]^2**	**Natural frequency ω [rad/s]**
1	$1.21 \cdot 10^7$	3482.50
2	$1.21 \cdot 10^7$	3482.50
3	$2.06 \cdot 10^7$	4539.13
4	$7.06 \cdot 10^7$	8401.54
5	$8.15 \cdot 10^7$	9030.12
6	$8.15 \cdot 10^7$	9030.12
7	$1.21 \cdot 10^8$	11003.47
8	$1.70 \cdot 10^8$	13030.49
9	$1.85 \cdot 10^8$	13592.62
10	$1.94 \cdot 10^8$	13922.06

Table 2 - Non-rotating cube results

3.2.2. 3D Cube. Rotating case. Fixed reference system

The study of the rotating case has to be carried out using the new state-space formulation (eq. 2.57), because the gyroscopic matrix appears. Comparing the results obtained with different velocities of rotation is useful. Thus, the results for a rotating cube are:

$\boldsymbol{\Omega = 50.00\ rad/s}$		
Mode number	**Eigenvalue $\lambda = i\omega$ [rad/s]**	**Natural frequency ω [rad/s]**
1	+3482.44 i	3482.44
2	-3482.44 i	3482.44
3	+3482.48 i	3482.48
4	-3482.48 i	3482.48
5	+4539.23 i	4539.23
6	-4539.23 i	4539.23
7	+8401.57 i	8401.57
8	-8401.57 i	8401.57
9	+9030.13 i	9030.13
10	-9030.13 i	9030.13
11	+9030.22 i	9030.22
12	-9030.22 i	9030.22
13	+11003.44 i	11003.44
14	-11003.44 i	11003.44
15	+13030.46 i	13030.46
16	-13030.46 i	13030.46
17	+13592.60 i	13592.60
18	-13592.60 i	13592.60
19	+13922.88 i	13922.88
20	-13922.88 i	13922.88

Table 3 - Rotating cube results in the fixed reference system (rotational speed = 50 rad/s*)*

$\Omega = 500.00$ rad/s		
Mode number	**Eigenvalue $\lambda = i\omega$ [rad/s]**	**Natural frequency ω [rad/s]**
1	+3480.56 i	3480.56
2	-3480.56 i	3480.56
3	+3476.50 i	3476.50
4	-3476.50 i	3476.50
5	+4538.71 i	4538.71
6	-4538.71 i	4538.71
7	+8403.84 i	8403.84
8	-8403.84 i	8403.84
9	+9031.51 i	9031.51
10	-9031.51 i	9031.51
11	+9039.66 i	9039.66
12	-9039.66 i	9039.66
13	+11000.00 i	11000.00
14	-11000.00 i	11000.00
15	+13027.80 i	13027.80
16	-13027.80 i	13027.80
17	+13590.48 i	13590.48
18	-13590.48 i	13590.48
19	+13904.85 i	13904.85
20	-13904.85 i	13904.85

Table 4 - Rotating cube results in the fixed reference system (rotational speed = 500 rad/s*)*

$\Omega = 1000.00$ rad/s		
Mode number	**Eigenvalue $\lambda = i\omega$ [rad/s]**	**Natural frequency ω [rad/s]**
1	+3474.80 i	3474.80
2	-3474.80 i	3474.80
3	+3458.80 i	3458.80
4	-3458.80 i	3458.80
5	+4537.50 i	4537.50
6	-4537.50 i	4537.50
7	+8409.50 i	8409.50
8	-8409.50 i	8409.50
9	+9035.56 i	9035.56
10	-9035.56 i	9035.56
11	+9069.30 i	9069.30
12	-9069.30 i	9069.30
13	+10099.99 i	10099.99
14	-10098.99 i	10099.99
15	+13019.68 i	13019.68
16	-13019.68 i	13019.68
17	+13584.03 i	13584.03
18	-13584.03 i	13584.03
19	+13862.17 i	13862.17
20	-13862.17 i	13862.17

Table 5 - Rotating cube results in the fixed reference system (rotational speed = 1000 rad/s*)*

$\Omega = 3482.50$ rad/s		
Mode number	**Eigenvalue $\lambda = i\omega$ [rad/s]**	**Natural frequency ω [rad/s]**
1	+3228.87 i	3228.87
2	-3228.87 i	3228.87
3	+3391.72 i	3391.72
4	-3391.72 i	3391.72
5	+4521.50 i	4521.50
6	-4521.50 i	4521.50
7	+8378.50 i	8378.50
8	-8378.50 i	8378.50
9	+9091.17 i	9091.17
10	- 9091.17 i	9091.17
11	+9584.55 i	9584.55
12	-9584.55 i	9584.55
13	+10840.96 i	10840.96
14	-10840.96 i	10840.96
15	+12892.28 i	12892.28
16	-12892.28 i	12892.28
17	+13490.47 i	13490.47
18	-13490.47 i	13490.47
19	+13589.61 i	13589.61
20	-13589.61 i	13589.61

Table 6 - Rotating cube results in the fixed reference system (rotational speed = first non-rotating natural frequency)

3.2.3. Results comparison. Non-rotating vs. Rotating cube

Observing the tables, it is easy to realize that the natural frequencies are so similar in the non-rotating case and in the rotating cases when the rotational speed is low. In the rotating case, as state-formulation is used, there are twice eigenvalues (the state matrix is a square matrix of the size 2·NDOF). It is important to note that rotating values appears twice because if a complex solution exists, its conjugate always exists. Despite this, it is important to notice that in the non-rotating case each eigenvalue corresponds to two natural frequencies, because:

$$\omega = \pm\sqrt{\lambda} \tag{3.6}$$

Hence, a parallelism between the mode numbers of the non-rotating case and the mode numbers of the rotating case can be made. Each two rotating values correspond to one non-rotating value. Thus, there is the same number of natural frequencies in all the tables.

As the rotational speed increases, the difference between the rotating natural frequencies and the non-rotating natural frequencies increases. When the rotational speed increases, the values of the natural frequencies become lower. The differences can be observed in the following table:

Natural frequency ω [rad/s]					
Natural frequency number	**$\Omega = 0$ rad/s**	**$\Omega = 50.00$ rad/s**	**$\Omega = 500.00$ rad/s**	**$\Omega = 1000.00$ rad/s**	**$\Omega = 3482.50$ rad/s**
1	3482.50	3482.44	3480.56	3474.80	3228.87
2	3482.50	3482.48	3476.50	3458.80	3391.72
3	4539.13	4539.23	4538.71	4537.50	4521.50
4	8401.54	8401.57	8403.84	8409.50	8378.50
5	9030.12	9030.13	9031.51	9035.56	9091.17
6	9030.12	9030.22	9039.66	9069.30	9584.55
7	11003.47	11003.44	11000.00	10099.99	10840.96
8	13030.49	13030.46	13027.80	13019.68	12892.28
9	13592.62	13592.60	13590.48	13584.03	13490.47
10	13922.06	13922.88	13904.85	13862.17	13589.61

Table 7 - Cube results in the fixed reference system

The following graph shows the variation of the first eigenvalue due to the rotational speed of the structure.

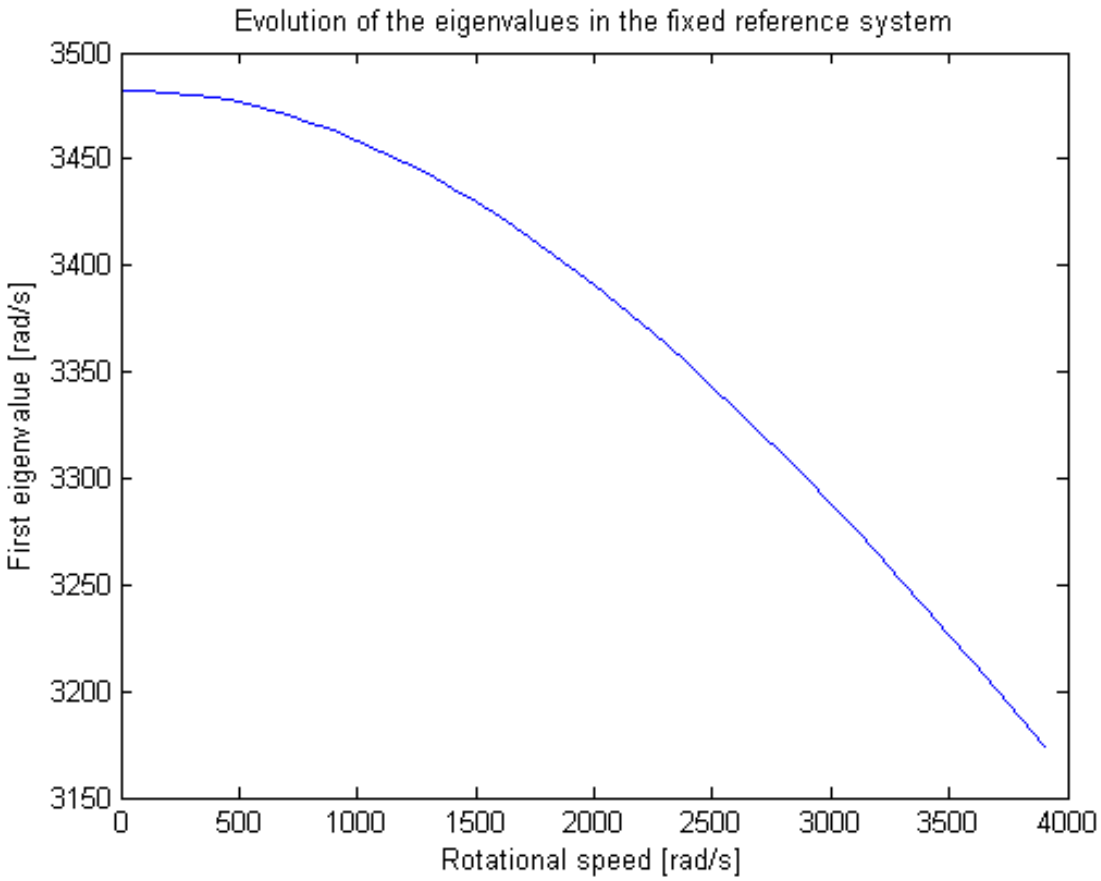

Figure 3 - Evolution of the eigenvalues in the fixed reference system

As the rotational speed increases, its effect in the eigenvalues also increases. When the rotational speed is low, the first eigenvalue is almost the same than in the non-rotating case (in velocities of rotation up to 250 rad/s).

The rotational speed affects the eigenvalues, and observing what happens as this velocity increases is interesting. Although rotational velocities higher than 4000 rad/s are not common, knowing their effect on the eigenvalue when they are high is useful. It is also important to notice that the first non-rotating eigenvalue is really high because of the material of the cube, which is steel. The following graph is an extension of the previous one, with the same analysis for rotational velocities up to 20000 rad/s.

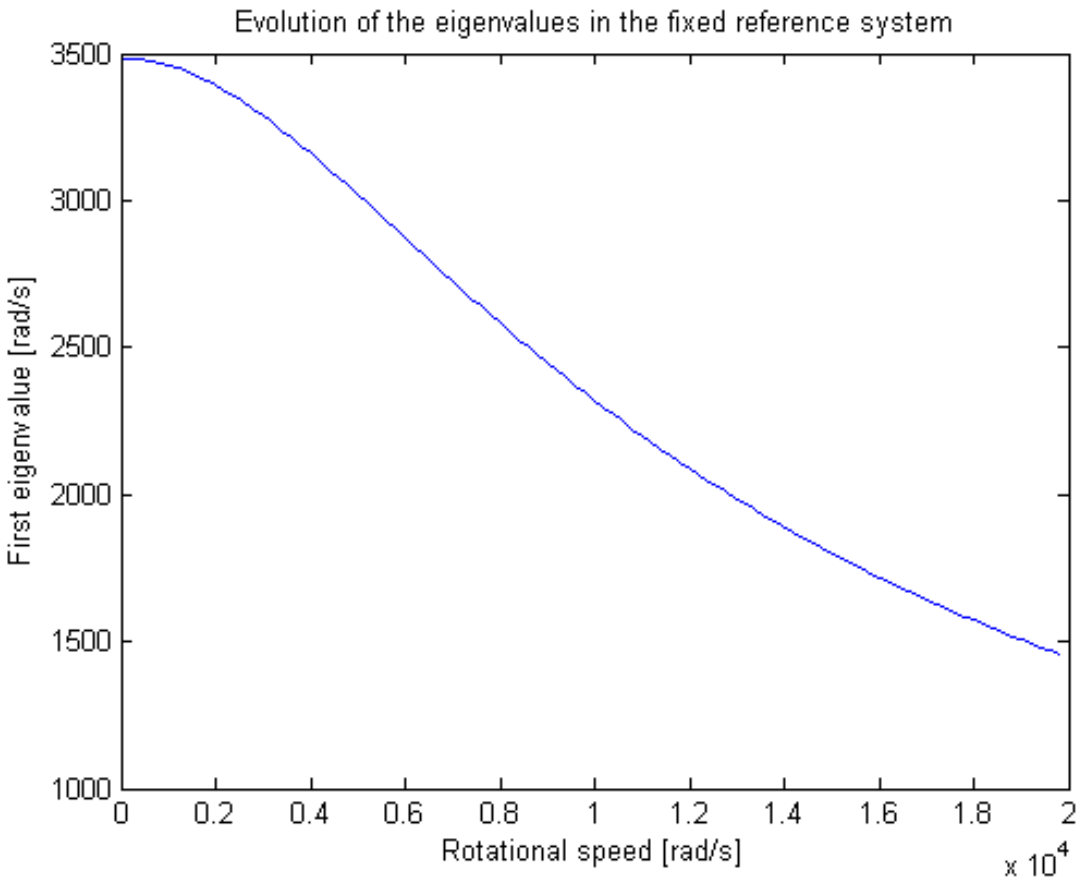

Figure 4 - Evolution of the eigenvalues in the fixed reference system (high rotational speeds)

The effect of the rotational speed follows the same tendency when it is higher than 4000 rad/s. The first eigenvalue, observed from the external point of view (fixed reference system) becomes lower as the rotational speed increases.

3.2.4. 3D Cube. Rotating case. Rotating reference system

The study in the rotating reference system shows, obviously, different results.

$\Omega = 50.00$ rad/s		
Mode number	**Eigenvalue $\lambda = i\omega$ [rad/s]**	**Natural frequency ω [rad/s]**
1	+3481.90 i	3481.90
2	-3481.90 i	3481.90
3	+3482.39 i	3482.39
4	-3482.39 i	3482.39
5	+4538.97 i	4538.97
6	-4538.97 i	4538.97
7	+8401.50 i	8401.50
8	-8401.50 i	8401.50
9	+9030.10 i	9030.10
10	-9030.10 i	9030.10
11	+9030.37 i	9030.37
12	-9030.37 i	9030.37
13	+11003.25 i	11003.25
14	-11003.25 i	11003.25
15	+13030.32 i	13030.32
16	-13030.32 i	13030.32
17	+13592.49 i	13592.49
18	-13592.49 i	13592.49
19	+13921.28 i	13922.28
20	-13921.28 i	13922.28

Table 8 - Rotating cube results in the rotating reference system (rotational speed = 50 rad/s)

$\Omega = 500.00$ rad/s		
Mode number	**Eigenvalue $\lambda = i\omega$ [rad/s]**	**Natural frequency ω [rad/s]**
1	+3471.98 i	3471.98
2	-3471.98 i	3471.98
3	+3422.96 i	3422.96
4	-3422.96 i	3422.96
5	+4523.83 i	4523.83
6	-4523.83 i	4523.83
7	+8395.06 i	8395.06
8	-8395.06 i	8395.06
9	+9055.80 i	9055.80
10	-9055.80 i	9055.80
11	+9028.16 i	9028.16
12	-9028.16 i	9028.16
13	+10981.48 i	10981.48
14	-10981.48 i	10981.48
15	+13013.31 i	13013.31
16	-13013.31 i	13013.31
17	+13579.37 i	13579.37
18	-13579.37 i	13579.37
19	+13856.94 i	13856.94
20	-13856.94 i	13856.94

Table 9 - Rotating cube results in the rotating reference system (rotational speed = 500 rad/s*)*

$\Omega = 1000.00$ rad/s		
Mode number	**Eigenvalue $\lambda = i\omega$ [rad/s]**	**Natural frequency ω [rad/s]**
1	+3248.31 i	3248.31
2	-3248.31 i	3248.31
3	+3438.99 i	3438.99
4	-3438.99 i	3438.99
5	+4479.75 i	4479.75
6	-4479.75 i	4479.75
7	+8359.21 i	8359.21
8	-8359.21 i	8359.21
9	+9022.32 i	9022.32
10	-9022.32 i	9022.32
11	+9144.37 i	9144.37
12	-9144.37 i	9144.37
13	+10915.43 i	10915.43
14	-10915.41 i	10915.41
15	+12961.03 i	12961.03
16	-12961.03 i	12961.03
17	+13539.83 i	13539.83
18	-13539.83 i	13539.83
19	+13730.25 i	13730.25
20	-13730.25 i	13730.25

Table 10 - Rotational cube results in the rotating reference system (rotational speed = 1000 rad/s*)*

$\Omega = 3482.50$ rad/s		
Mode number	**Eigenvalue $\lambda = i\omega$ [rad/s]**	**Natural frequency ω [rad/s]**
1	+95.25 i	95.25
2	-95.25 i	95.25
3	+2372.43 i	2372.43
4	-2372.43 i	2372.43
5	+4089.88 i	4089.88
6	-4089.88 i	4089.88
7	+7371.67 i	7371.67
8	-7371.67 i	7371.67
9	+8936.18 i	8936.18
10	- 8936.18 i	8936.18
11	+9964.09 i	9964.09
12	-9964.09 i	9964.09
13	+10429.57 i	10429.57
14	-10429.57 i	10429.57
15	+12126.71 i	12126.71
16	-12126.71 i	12126.71
17	+12920.20 i	12920.20
18	-12920.20 i	12920.20
19	+12969.71 i	12969.71
20	-12969.71 i	12969.71

Table 11 - Rotational cube results in the rotating reference system (rotational speed = first non-rotating natural frequency)

The following graph shows the variation of the first eigenvalue due to the rotational speed of the structure.

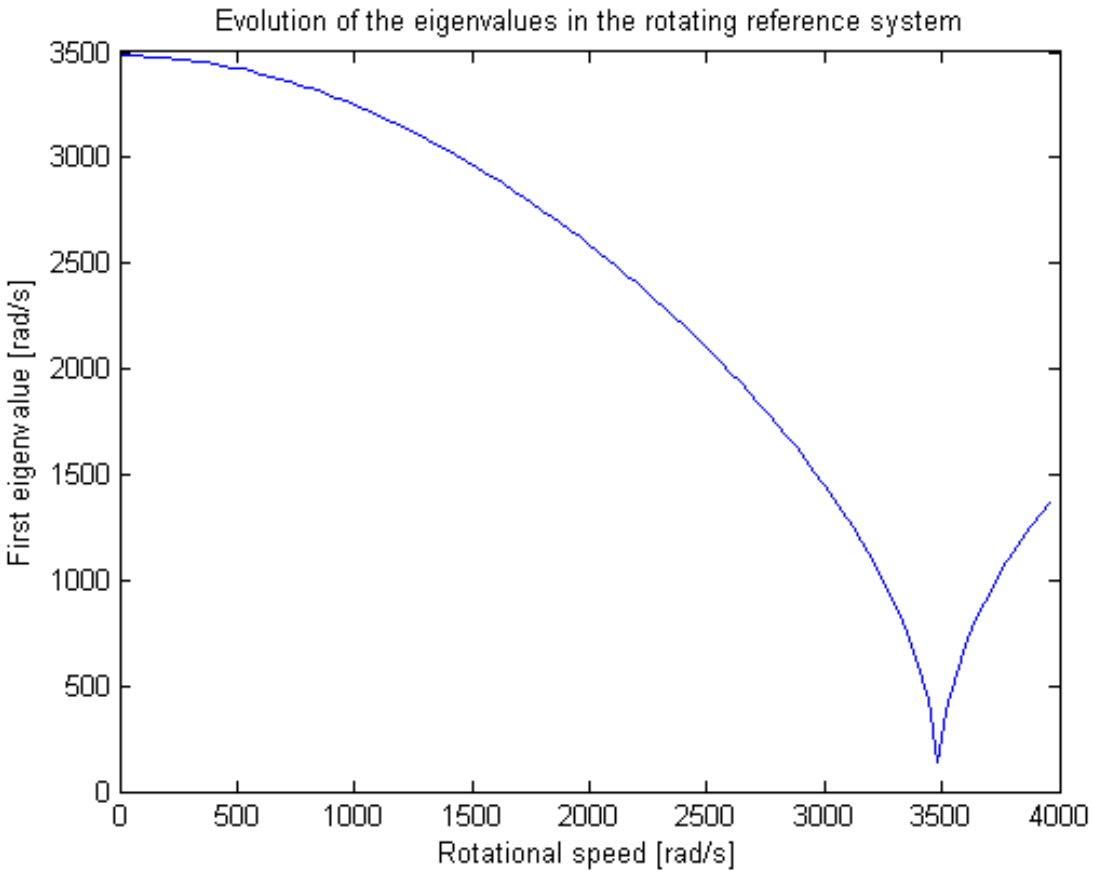

Figure 5 - Evolution of the eigenvalues in the rotating reference system

As the rotational speed increases, its effect in the reduction of the eigenvalues also increases. There is a singularity when the rotational speed of the structure is equal to the first eigenvalue in the non-rotating case, which is 3482.50 rad/s. The rotational speeds that coincide with natural frequencies are called critical speeds.

This phenomenon will appear again in higher velocities of rotation, which coincide with other natural frequencies of the non-rotating case. Although higher velocities of rotation are not common (a car engine rotates at more or less 4000 rpm, which means 418.9 rad/s), their study is useful in order to observe the behavior of the eigenvalues. The following graph is an extension of the previous one, with the same analysis for rotational velocities up to 20000 rad/s.

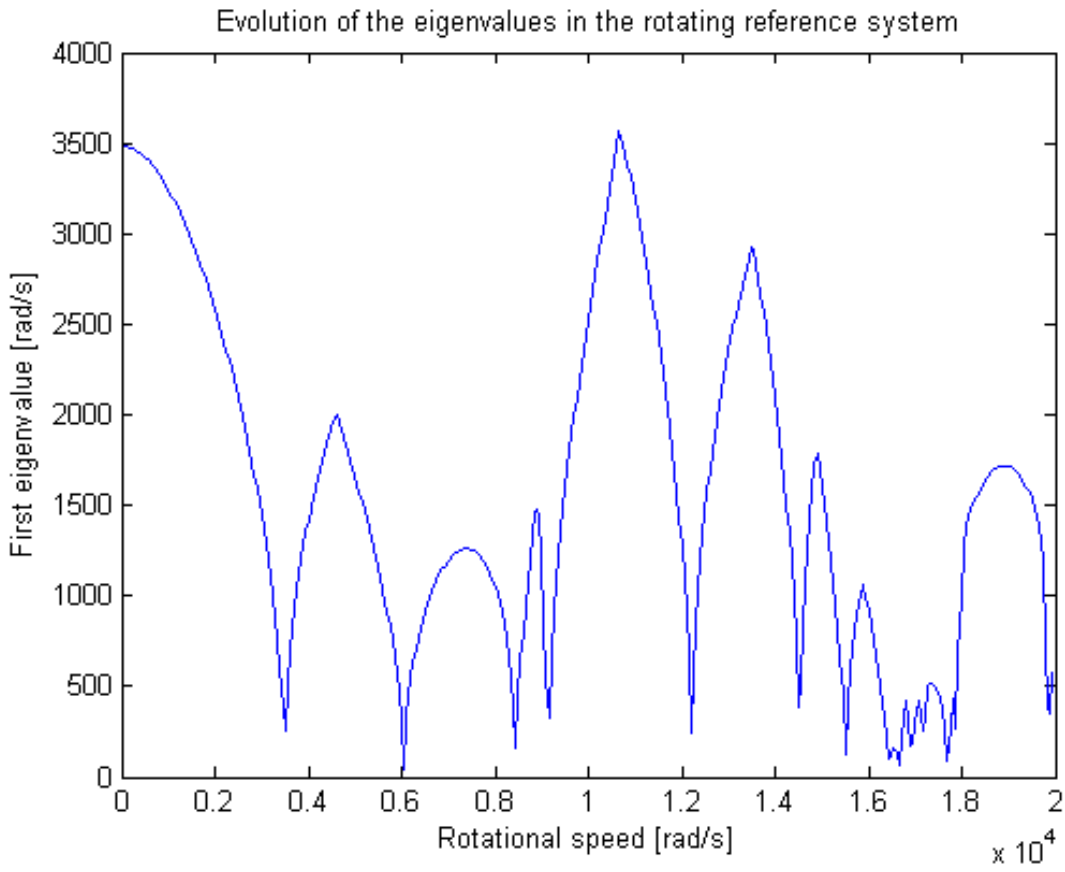

Figure 6 - Evolution of the eigenvalues in the rotating reference system (high rotational speeds)

There are some low peaks that coincide with eigenvalues in the non-rotating cube (table 2). This is the case, for example, of the peaks in 3842.50 rad/s, in 8401.54 rad/s and 9030.12 rad/s. On the other hand, there are some high peaks that also coincide with eigenvalues in the non-rotating cube. This is the case of the peaks in 4539.13 rad/s and 11003.47 rad/s.

The low peaks are supposed to be 0 rad/s. This means that when the structure is rotating at a rotational speed equal to a natural non-rotating frequency, its movement becomes infinite. The eigenvalues are not exactly equal to zero because the code has some inherent errors, due to FEM formulation and MATLAB approximations. Moreover, the graph has been created using steps of 50 rad/s for the rotational speed in order to calculate the corresponding eigenvalue. The graph would be more accurate if the steps were smaller, but it means much computation time. This discretization is useful in order to have enough accurate results without too much computation time. The high peaks represent changes in the tendency of the eigenvalues.

3.3. Future work

There are some capabilities that would be added to the code. First of all, analyzing the same structures with damping (and rotation) could be interesting. The equation changes if damping is added, but the most difficult part would not be the new formulation, it would be the modeling of this damping, because finding its right values can be really difficult.

The most important step in the future is the analysis of noise radiation of these rotating structures. This will be a long process, because there are a lot of parameters to take into account. First of all, the study now provides the eigenvalues and eigenvectors of rotating structures, not their noise radiation value. Secondly, the influence between the structures and the environment that surrounds them is very important. The properties of this environment affect the noise propagation. Besides, the environment can modify the structure response: the movement of the structure creates aerodynamic forces, and these aerodynamic forces interact with the structure. Hence, this step is really complicated, and it will be reached using BEM, because it is a more efficient method for this kind of studies.

The creation of this general code, making it generic (it means that the user can introduce the structure properties and geometry that he wants and the environment characteristics that he wants), can last more than 6 months, if only one researcher works on it. This code would be very useful for the study of noise radiation for a lot of real cases, such as an engine, a turbine, a railway axis… After studying the noise radiation of these structures, some solutions for the noise problem can be achieved, and the information given by the code will be essential. This code will be useful for solving existing problems, but it will be also useful for predicting issues before constructing the structures. Predicting the problems is almost always cheaper than solving an existing problem, so this code would be essential in the design of structures with critical noise radiation characteristics.

Before all these steps, a comparison between the numerical results and some analytical results should be done. There is a very interesting paper that analyzes the dynamic behavior of a hollow cylinder, and its results can be used in order to check the code.

3.3.1. Analytical analysis

It is difficult to find analytical eigenvalues in rotating structures. Despite this, the *Journal of Sound and Vibration* published an article (Sun, Chu, & Cao, 2012) that analyzes thin rotating cylindrical shells, and their results in natural frequencies are reliable.

Researchers Shupeng Sun, Shiming Chu and Dengqing Cao, from the School of Astronautics in Harbin Institute of Technology, China, have studied the vibration characteristics of this kind of structures. A cylindrical shell can be considered thin when its length is higher than 10 times its thickness. They use sets of Fourier series, which consist of long cosine and sine developments, and the Stokes transformation for the expressions of the displacement field, in order to find exact solutions for thin rotating cylindrical shells.

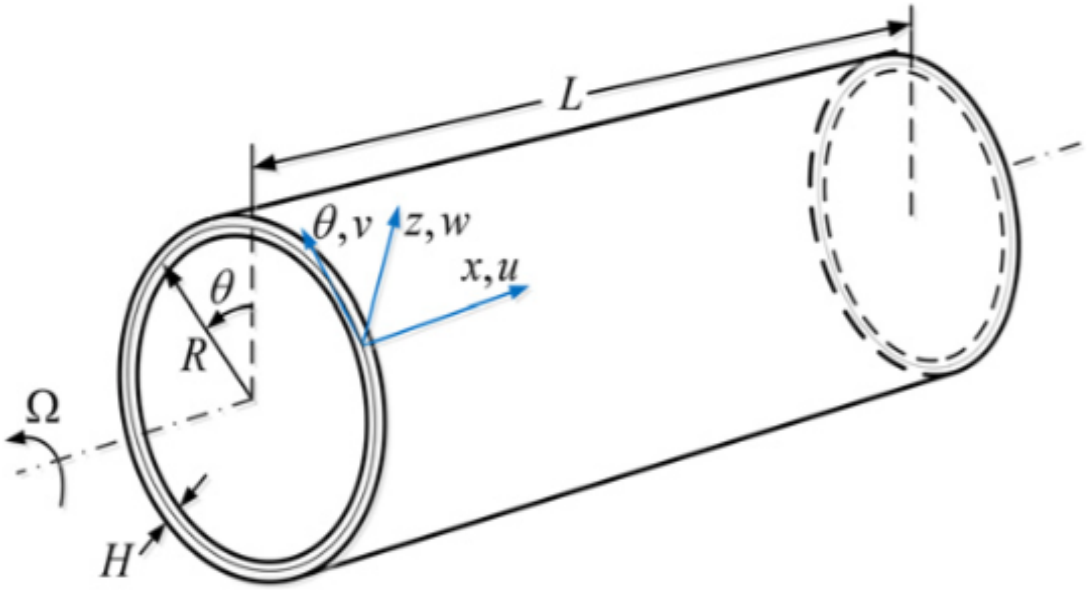

Figure 7 - Cylinder geometry (Sun, Chu, & Cao, 2012)

This paper provides analytical solutions for thin rotating cylindrical shells with classical boundary conditions, such as clamped-clamped (C-C), clamped-free (C-F), clamped-simply supported (C-S) and simply supported-simply supported (S-S). These boundary conditions express how are the cylinder ends subjected. This paper uses the notation:

R	Mean radius of the cylinder [m]
H	Thickness of the cylinder [m]
L	Length of the cylinder [m]
E	Young's modulus [Pa]
$\boldsymbol{\nu}$	Poisson's ratio
$\boldsymbol{\rho}$	Material density [kg/m^3]
$\boldsymbol{\omega}$	Natural frequency [rad/s]
$\boldsymbol{\Omega}$	Rotational speed of the structure [rad/s]
n	Mode number

Table 12 - Analytical notation

and for the analysis of the results it uses non-dimensional parameters, indicated by the superscript *. These non-dimensional parameters are

L/R	Length/mean radius ratio
H/R	Thickness/mean radius ratio
$\boldsymbol{\omega}^*$	Non-dimensional natural frequency
$\boldsymbol{\Omega}^*$	Non-dimensional rotational speed of the structure

Table 13 - Non-dimensional parameters

where the non-dimensional natural frequency and non-dimensional rotational speed of the structure are, respectively,

$$\omega^* = \gamma\omega \tag{3.7}$$

$$\Omega^* = \gamma\Omega \tag{3.8}$$

and γ is

$$\gamma^2 = \frac{\rho R^2(1-\nu^2)}{E} \quad \rightarrow \quad \gamma = \sqrt{\frac{\rho R^2(1-\nu^2)}{E}} \ . \tag{3.9}$$

They have checked their expressions calculating the natural frequencies of some structures with $\Omega^* = 0$, which means that the cylinder shell is not rotating, and they also compare their results with the results obtained in the study of the vibrations in finite long rotating cylinders made by other researchers (Saito & Endo, 1986).

3.3.1.1. PARTICULAR CASES

The analytical study done by *Cao et al.* of the eigenvalues is complicated, because the results depend on the number of terms of the Fourier series. However, the results converge when the number of terms increases, and these researchers have demonstrate that it is not necessary to use more than 800 or 1000 terms, because from this point the results only varies from the fifth or the sixth decimal.

They analyzed a thin rotating cylindrical shell with boundary conditions C-C and the characteristics:

L/R	10
H/R	0.05
$\boldsymbol{\nu}$	0.3

Table 14 - Non-dimensional characteristics of the cylinder in the analytical case

For rotating cylindrical shells, travelling waves (backward waves ω_b and forward waves ω_f) instead of standing waves are investigated, because Coriolis acceleration (which is induced by rotation) bifurcates the shells frequencies. Although this, the values of backward and forward waves are very similar if the rotational speed is low.

Cao et al. results ($\boldsymbol{\Omega^* = 0.0025}$)		
Mode number n	$\boldsymbol{\omega_b^*}$	$\boldsymbol{\omega_f^*}$
2	0.05996	0.05596
3	0.11457	0.11156
4	0.21316	0.21079
5	0.34227	0.34032

Table 15 - Cao et al. results (rotational non-dimensional velocity = 0.0025)

Cao et al. results ($\boldsymbol{\Omega^* = 0.0050}$)		
Mode number n	$\boldsymbol{\omega_b^*}$	$\boldsymbol{\omega_f^*}$
2	0.06228	0.05428
3	0.11662	0.11060
4	0.21493	0.21019
5	0.34385	0.33996

Table 16 - Cao et al. results (rotational non-dimensional velocity = 0.0050)

3.3.1.1. EQUATION CONSTRUCTION

It is important to notice that the analytical studies takes into account Coriolis terms, so it has been done using the rotating reference system (2.56).

The study of the dynamic response in the rotating coordinate system is interesting in order to compare the results with the analytical ones. However, the study of the dynamic response in the fixed system is also important and interesting, because it gives information of the external behavior of the structures. This information is essential for the future work, the analysis of noise radiation of rotating structures.

3.3.1.2. PARTICULAR CASE

In order to compare the results with the analytical ones, the material properties and the geometry of the cylinder should be the same. Therefore, according to table 10, the chosen geometry could be:

R	1 m
H	0.05 m
L	10 m

Table 17 - Geometric characteristics of the cylinder

The Poisson's ratio must be the same as the one of the analytical case. Hence, selecting typical material properties for a cylinder made by steel (Steel Construction, 2012), the rotational speed is determined (using equation 3.9):

$\boldsymbol{\rho}$	7850 kg/m^3
E	2.07·10^{11} Pa
$\boldsymbol{\nu}$	0.3

Table 18 - Material properties of the cylinder

$$\gamma = \sqrt{\frac{\rho R^2(1-\nu^2)}{E}} = \sqrt{\frac{7850 \cdot 1^2(1-0.3^2)}{2.07 \cdot 10^{11}}} = 1.858 \cdot 10^{-4}\ \text{s}$$

It is necessary to compare the results in two different cases.

$$\begin{cases} \Omega^* = 0.0025 \\ \Omega^* = \gamma\Omega \end{cases} \rightarrow \Omega = 13.46\ \text{rad/s} \tag{3.10}$$

$$\begin{cases} \Omega^* = 0.0050 \\ \Omega^* = \gamma\Omega \end{cases} \rightarrow \Omega = 26.92\ \text{rad/s} \tag{3.11}$$

3.3.1.1. HOLLOW CYLINDER CONSTRUCTION

In order to obtain the results using the code, the creation of a hollow cylinder will be necessary. The first option in order to achieve this task would be the creation of this geometry using GiD, after loading the "problem type" called 3DHexahedral, which was made by researcher Alejandro de Haro. This "problem type", as it has been explained previously, extracts the information of the nodes, elements, materials… in a file text with a specific format. The existing MATLAB code is able to read this information and to transform it to structured matrices, which provides all the information for the creation of the stiffness and mass matrices in order to solve the problem.

Hence, a hollow cylinder would be created using GiD. After the creation of the cylinder, it will have to be meshed with hexahedral elements, because FEM notation in MATLAB code uses the shape functions and the processes from hexahedral elements. The problem will come from the GiD mesh creator, because it cannot build mesh using hexahedral elements in a cylinder (GiD is only able to create a mesh using tetrahedral elements in this kind of structures). In this situation, there will be two options. The first one is changing the existing code in order to make it able to accept tetrahedral mesh elements. The second option is the use of another preprocessor, like PATRAN.

The first option implies a big work, because tetrahedral and hexahedral mesh elements have different processes and shape functions. Virtually all the existing code would have to be remade if this option is chosen, because the process of extracting data, the process of integration and the building of the problem matrices would be different of the existing ones.

The second option implies building another hollow cylinder in a preprocessor able to use hexahedral mesh elements in these structures. The main problem of this option is the fact that there is not a "problem type" created for another preprocessor. This means that data extracted from another preprocessor will not have the needed format, so the existing MATLAB code will not be able to read these data and to interpret them. Despite this, extracting the information of the data in a text file is possible using the preprocessor PATRAN, which is one of the most useful preprocessors in this situation. Hence, a new work will appear: the data processing in order to introduce it to the code.

This work would be achieved using MATLAB, extracting the information from the text file and creating matrices.

3.4. Temporal aspects

The development of the code for the analysis of rotating structures has lasted approximately 3 months. The most difficult part in the project has been the understanding of the theoretical foundation, because it is really complicated. This process has taken more than 2 months. The understanding of the performance of the existing code has lasted two weeks approximately.

The development of the next part of the code, in order to study the noise vibration of rotating structures, would last a lot of time. As it has been explained in the previous section, it can last more than 6 months. The processes that will have to be carried out are explained in the previous section.

The study of Boundary Element Method techniques would last 1 month, and the modification of the existing code would last at least 2 or 3 months. The interaction with aerodynamic forces, the aerodynamics noise and other problems of the surrounding environment are really complicated topics, so their solution can last a lot of time. After programming the code, it should be validated through analytical cases or experimental cases. This process would last more than a month.

3.5. Budget

This project has consisted on the development of a MATLAB code, so there are some costs that have to be taken into account. Briefly, the costs of developing this code have been:

CONCEPT		Total cost	Cost in the project
Software	MATLAB	2000.00 €	133.33 €
	PATRAN	10000.00 €	175.98 €
	GiD	1540.00 €	17.72 €
Computer		800.00 €	66.67 €
Electricity		5.54 €	5.54 €
Researcher		4500.00 €	4500.00 €
TOTAL		**18845.54 €**	**4899.24 €**

Table 19 - Budget

The details are attached on the document “Budget”.

3.6. Accomplishment of the requirements

The first requirement was that the new MATLAB code must be able to read the information given by the preprocessor. This aspect has been accomplished in some structures, such as cubes. The fact that GiD is not able to mesh curved structures with hexahedral elements forces the user to use another preprocessor, PATRAN. The information given by PATRAN has not the same format as the information given by GiD, so MATLAB cannot read it. This problem has been solved processing data with some MATLAB functions.

The second requirement was that the code must be able to solve the dynamic response of rotating structures, especially eigenvectors and eigenvalues. The creation of new matrices and the changing of the solving process have achieved this requirement. The user can choose the rotational speed of the structure, and the axis of rotation is the z-axis.

The third requirement was that the code must be efficient computationally. All the matrices are of the kind “sparse”, which saves a lot of memory in matrices with a lot of zeros. Despite this, in some cases the number of nodes (and therefore the number of DOF) is so huge that the code is working during a lot of time.

The last requirement was that the code must adapt to the needs of LEAM. As it can be seen in the annex, the new code takes advantage of a lot of existing functions, and the performances of the new functions are explained as MATLAB comments. The code is able to calculate what LEAM needs, and their researchers will be able to modify this code easily.

3.7. Conclusions

The study of the dynamic response of rotating structures is not simple. It comes from a strong and complicated theoretical foundation. Applying this theoretical foundation in a FEM based solver is neither easy. The relation between the pre-processor and MATLAB can be an issue, and the creation of a mesh almost always causes problems. The main problems using MATLAB are related to the use of RAM memory. When MATLAB works with so huge matrices, the code must be the most efficient possible. This means avoiding complicated operations, iterative processes, inverting matrices or building “full” matrices.

The natural frequencies in structures vary very little when they are rotating with relatively low rotational velocities and when they are not rotating. This difference between natural frequencies increases when the rotational speed becomes higher, but the effect is only appreciable when the rotational speed is higher than about 500 rad/s in the studied case, which means very high velocity. The rotational speed of an engine of a Formula 1 car is about 800 rad/s, so there would be effects in some structures due to the rotation. The rotational speed makes natural frequencies to be lower.

Natural frequencies are high because of the material of the cube, which is a kind of steel. The fact that it has high Young’s Modulus value and high density makes eigenvalues to be high. Other structures could have lower eigenvalues, depending on their material properties and their geometry.

Although the gyroscopic matrix appears in the place of the damping matrix in the state space formulation, their effects are not the same. First of all, the eigenvalues found in rotating structures without damping are pure complex numbers. This means that there is no reduction of the amplitude of the vibration, because there is not a real part. Contrarily, the damping causes a reduction of the amplitude of the vibration through time, and this means that the eigenvalues in structures with damping have real part. Moreover, the effect of the rotation in the eigenvalues is small unless the rotational speed is really high, and the effect of damping could be very important regardless of the rotational speed.

In the fixed reference system the eigenvalues decreases as the rotational speed increases. It means that the first natural frequency becomes lower when the rotational speed is high, and this would be very important in the study of noise radiation in structures with high rotational speeds. On the other hand, the effect of the rotational speed in the rotating reference system is not so clear. It produces peaks in the eigenvalues when it is equal to some natural non-rotating frequencies, and it can produce issues in the structures. Despite this, the study of the response in the fixed reference system is the most important study because the future code will analyse noise propagation, so the external point of view is needed.

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